UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Advanced Mechanics I (Phys. 709) — Prof. Leo C. Stein — Fall 2018

Problem Set 2

Due: Friday, Sep. 7, 2018, by 5PM

I'm attempting to perform a binary search in the space of amount of homework to assign, so there are far fewer problems this week than last week. As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. FW1.11

2. **Angular momentum under Galilean invariance**. For this problem, we will consider a system of N particles in 3 dimensions, with no external forces. Let the Lagrangian be of the form

$$L = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{\mathbf{r}}}_i^2 - \sum_{i \neq j} V_{ij}(r_{ij}) \,, \tag{1}$$

where recall that $\vec{\mathbf{r}}_{ij} \equiv \vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j$, and $r_{ij} = |\vec{\mathbf{r}}_{ij}| = \sqrt{\vec{\mathbf{r}}_{ij} \cdot \vec{\mathbf{r}}_{ij}}$.

   (a) Find the variational derivatives

$$\frac{\delta r_{ij}}{\delta \vec{\mathbf{r}}_i} \,, \quad \frac{\delta r_{ij}}{\delta \vec{\mathbf{r}}_j} \,, \quad \text{and} \quad \frac{\delta r_{ij}}{\delta \vec{\mathbf{r}}_k} \quad (k \neq i \,, k \neq j) \,. \tag{2}$$

   Recall that the variational derivative δ acts like a "derivation" and thus satisfies a chain rule and product rule.

   (b) What are the equations of motion for the ith particle? What can you note about the force on the ith particle due to the jth particle, as compared to the force on the jth particle due to the ith particle?

   (c) What is the total angular momentum of the mechanical system, about the origin?

   (d) What is the time derivative of the total angular momentum?

3. **Order of solving/varying**. Let's look at a 2-body central force problem, now already reduced to spherical polar coordinates, with the Lagrangian being

$$L = \frac{1}{2} \mu \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \, \dot{\phi}^2 \right) - V(r) \,. \tag{3}$$

Recall that the p_ϕ momentum, traditionally called ℓ, is constant,

$$\mu r^2 \sin^2 \theta \, \dot{\phi} = \ell = \text{const.} \tag{4}$$

   (a) What is the conserved energy E associated with the Lagrangian in Eq. (3)?

   (b) Eliminate $\dot{\phi}$ from E by using the conserved ℓ from Eq. (4). Now identify the effective potential $V_{\text{eff}}(r)$ (assume that the particle orbits in the x–y plane).

   (c) Now try the opposite order. Try to eliminate $\dot{\phi}$ from L by using the conserved ℓ from Eq. (4). Does this give the same effective potential as before?

   (d) Can you explain why? (Hint: variation of the action must be performed by letting the generalized coordinates vary independently)

4. **Computer tool practice** (not for credit). Computers are crucial research tools for physics nowadays. This exercises are not for credit because this is not a programming or numerical methods course, and since everyone has a different amount of experience with programming and numerics, it would be unfair to grade this work for credit. However I still encourage everyone to try this exercise.

   Extra feel-good points (not redeemable for credit) for making your code easy to understand, well documented, or even using a revision control system (e.g. git) to keep track of your code.

   (a) Using Mathematica, python, or your favorite programming environment: Generate a contour plot of energy in the θ–$\dot{\theta}$ plane for the rigid pendulum, whose Lagrangian (not energy!) is

   $$L = \frac{1}{2}\dot{\theta}^2 - \cos\theta\,. \tag{5}$$

   Here we have adjusted the units so that $mL^2 = 1$ and $mgL = 1$ (convince yourself that this is possible).

   Make sure that the following features are visible:

   i. The periodicity in θ
   ii. The separatrix
   iii. A fixpoint at the bottom of the potential, and
   iv. Nicely spaced curves demonstrating the different types of motion (circulating vs. oscillating).

   (b) In class we discussed how to find the period (and thus frequency) of bound motion in a potential, which involves integrating

   $$T = 2\sqrt{\frac{m}{2}}\int_{x_-}^{x^+}\frac{dx'}{\sqrt{E-V(x')}}\,, \tag{6}$$

   where $x_\pm$ are the turning points of the bound motion for a given energy E.

   The rigid pendulum above is a type of *anharmonic* oscillator, since the frequency of oscillation depends on the amplitude or energy. For this system (letting $m = L = g = 1$), compute the period or frequency for any bound oscillation. You are free to use numerical tools such as Mathematica's `NIntegrate`, or the python package `scipy`. Extra feel-good points for making a plot of period (or frequency) as a function of energy (or amplitude of oscillation). Also extra feel-good points if you get an analytical result which can be evaluated with specialized methods, instead of general integration methods. Super extra feel-good points for implementing your own numerical quadrature routine to perform the integration.