**UNIVERSITY OF MISSISSIPPI**

Department of Physics and Astronomy

Advanced Mechanics I (Phys. 709) — Prof. Leo C. Stein — Fall 2018

## Problem Set 3

**Due**: Monday, Sep. 17, 2018, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. **Hanging cable**. Consider a cable which can not stretch, so each arc of length $ds$ has a fixed mass density, $dm = \rho ds$. Let the cable be hanging so that its curve is the graph of $z(x)$).
   (a) What is the infinitesimal arc length $ds$ at some point $(x, z)$, in terms of the slope? What is the total length?
   (b) Let the cable be hanging between $-a \le x \le +a$, in a uniform gravitational field pointing up along $z$. What is the total potential energy of the cable?
   (c) Treat the total potential as an energy functional, and add an appropriate Lagrange multiplier to enforce that the length is constant. Now vary this functional to find a differential equation that $z(x)$ must satisfy.
   (d) (Extra credit) Integrate the diffeq so as to find the functional form of a hanging cable.

2. **Justifying your geometric intuition**. Take a cable again of length $L > 2a$ fixed in the $x$–$y$ plane at $(-a, 0)$ and $(+a, 0)$. Consider the area bounded by the cable and the $x$ axis, see Fig. 1. Write down an appropriate functional with Lagrange multiplier in order to maximize this area, subject to the constraint that the length must be constant. What does your intuition say the shape should be? Does the solution agree?

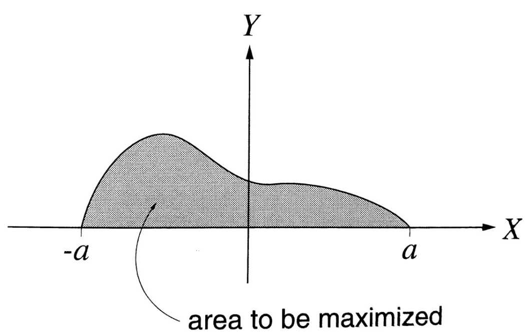


Figure 1: Area to be maximized

3. **Properties of orthogonal matrices**. Recall that an $n \times n$ matrix $\boldsymbol{U}$ is called orthogonal if it satisfies $\boldsymbol{U}^T\boldsymbol{U} = \mathbf{1}$, the identity matrix.
   (a) Prove that $\det(\boldsymbol{U}) = \pm 1$ (a rotation with positive determinant is called "proper," and with negative determinant "improper"; but that is just a composition of a proper rotation and a reflection).
   (b) For a proper rotation in any odd dimension $n$, prove that the orthogonal matrix $\boldsymbol{U}$ has at least one eigenvalue equal to 1; hence there is an "axis" of rotation – a direction that is invariant under the transformation $\boldsymbol{U}$. (Hint: first prove that if there is an eigenvalue equal to 1, then $\det(\boldsymbol{U} - \mathbf{1}) = 0$.)

(c) The trace of a matrix $\boldsymbol{A}$ is $\text{tr}(\boldsymbol{A}) = \sum_i A_{ii}$, the sum of the diagonal elements. Prove that the trace is invariant under a similarity transformation by an orthogonal matrix, i.e. that if $\boldsymbol{A}' = \boldsymbol{U}^T \boldsymbol{A} \boldsymbol{U}$, then $\text{tr}(\boldsymbol{A}') = \text{tr}(\boldsymbol{A})$. (Hint: first prove that $\text{tr}(\boldsymbol{AB}) = \text{tr}(\boldsymbol{BA})$.)

(d) **Special case of** $3 \times 3$. If $\mathbf{U}$ is a $3 \times 3$ orthogonal matrix, show that its trace is $\text{tr}(\boldsymbol{U}) = 1 + 2\cos\phi$, where $\phi$ is the angle of rotation.

4. **What is this matrix**? Consider the matrix $A$ with components

$$A_{jk} = (1 - \cos\phi)\hat{n}_j \hat{n}_k + \cos\phi\, \delta_{jk} - \sin\phi\, \hat{n}_i \epsilon_{ijk} \tag{1}$$

with $\hat{n}_i$ the components of a unit vector $\hat{n}$, $\epsilon_{ijk}$ the Levi-Civita symbol, and I am using the Einstein summation convention for repeated indices. The following identities are useful.

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{2}$$

$$\epsilon_{ijk} n_j n_k = 0. \tag{3}$$

(a) Show that $A$ is an orthogonal matrix, and so can be considered to be a rotation matrix.

(b) Write down the matrices $A$ for (i) $\hat{n} = \hat{x}$, (ii) $\hat{n} = \hat{y}$, and (iii) $\hat{n} = \hat{z}$.

(c) Show that $\text{tr}(A) = 1 + 2\cos\phi$.

(d) Show that a vector parallel to $\hat{n}$ is *not* changed by the rotation corresponding to $A$.

(e) From parts (a)-(d) we can argue that $A_{jk}$ is the rotation matrix for a rotation with axis $\hat{n}$ and rotation angle $\pm\phi$. Make some simple choice for $\hat{n}$ to determine whether the plus or minus sign should be used.