UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Advanced Mechanics I (Phys. 709) — Prof. Leo C. Stein — Fall 2018

Problem Set 4

Due: Wednesday, Sep. 26, 2018, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. **Commutators from Lie groups to Lie algebras**. Let G be some Lie group, which is not necessarily commutative, for example a group of matrices. The associated Lie algebra, $\mathfrak{g}$, is the space of infinitesimal transformations in the neighborhood of the identity element $\mathbb{1}$. That is, if we take the group element $A \in G$ given by $A = \mathbb{1} + \epsilon a$,

$$a = \left. \frac{d}{d\epsilon} A \right|_{\epsilon=0} \tag{1}$$

then $a \in \mathfrak{g}$ is in the Lie algebra, and we say that a "generates" A.

   (a) Find the inverse element A^{-1} up to order ϵ^2.

   (b) The commutator of two group elements $A, B \in G$ is given by the (non-commutative) product $ABA^{-1}B^{-1}$. Take A, B to be generated by a, b respectively. Expand out the commutator $ABA^{-1}B^{-1}$ up to order ϵ^2.

2. **A few moments of inertia**.

   (a) Consider a rectangular prism of uniform density ρ with side lengths a, b, c, centered at the origin, aligned with the (x, y, z) axes. Compute the moment of inertia tensor I_{ij}.

   (b) Now suppose that we rotate the shape in the x–y plane by 45°. What are two different ways to compute the moment of inertia tensor of the rotated prism? What is the new tensor $I_{i'j'}$?

   (c) Consider an *oblate spheroid* of uniform density D, with its principal axes aligned along x, y, z. This is the region that satisfies the inequality

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b} \leq 1\,, \tag{2}$$

   for axes $a > b$ (make $a < b$ if you want it to be prolate). Before calculating anything, make a guess about what the moment of inertia tensor will look like in these coordinates. Now compute its moment of inertia tensor and see if your intuition was correct. Hint: This is probably easiest by writing down the xyz integrals and then changing the integration variables to cylindrical coordinates (z, ρ, ϕ), where as usual $x = \rho \cos\phi$ and $y = \rho \sin\phi$. Of course don't forget the Jacobian factor, and choose the order of integration wisely to make your life as easy as possible.

3. **Surface of a spun-cast mirror**. One way to make a mirror is as follows. Sit a cylindrical vat on a turntable, so that the cylinder is spun around its axis, which is vertical ($\hat{z}$). Let this turntable spin at a frequency ω. Fill this cylinder with molten glass. The rotation of the cylinder couples to the viscous molten glass, making it spin, and it ends up with a curved surface. Now allow the glass to cool slowly, so that it solidifies with the curved surface, which is later given a reflective coating.

   Ignore the rotation of the Earth, and treat gravity as uniform in the $\hat{z}$ direction. Find the parametric form of the surface of the mirror. Hint: once the fluid glass has come into equilibrium, in the rotating frame, none of the fluid elements are moving; what does that mean about the potential difference (which potential?) between different surface fluid elements in the rotating frame?

4. **(In)stability of axes in torque-free precession**. Recall that precession is governed by Euler's equations,

$$N_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 \tag{3}$$
$$N_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 \tag{4}$$
$$N_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 \tag{5}$$

which have been evaluated in a body frame which also diagonalizes the moment of inertia tensor (so that $I_{ij} = \text{diag}(I_1, I_2, I_3)$ with $I_1 > I_2 > I_3$), and N_i are the components of external torque in this same body frame. We already saw that when external torques vanish, if the vector $\vec{\omega}$ is aligned with any of the three principal axes, then $\dot{\vec{\omega}} = 0$.

Choose $\vec{\omega}$ along the 1 axis, so $\vec{\omega} = (\omega_1, 0, 0)$. Now suppose we move slightly away from this solution, taking

$$\vec{\omega} = (\omega_1, 0, 0) + \epsilon(\delta\omega_1(t), \delta\omega_2(t), \delta\omega_3(t)) + \mathcal{O}(\epsilon^2)\,. \tag{6}$$

(a) Write out Euler's equations for this ansatz. Neglect terms of $\mathcal{O}(\epsilon^2)$.

(b) You should find that the two equations governing $\delta\omega_2(t)$ and $\delta\omega_3(t)$ are coupled to each other. Take a time derivative of each equation and decouple them.

(c) What is the general solution for $\delta\omega_{2,3}(t)$?

(d) Now repeat supposing we started with

$$\vec{\omega} = (0, \omega_2, 0) + \epsilon(\delta\omega_1(t), \delta\omega_2(t), \delta\omega_3(t)) + \mathcal{O}(\epsilon^2)\,, \tag{7}$$

and again with

$$\vec{\omega} = (0, 0, \omega_3) + \epsilon(\delta\omega_1(t), \delta\omega_2(t), \delta\omega_3(t)) + \mathcal{O}(\epsilon^2)\,. \tag{8}$$

Which axes are stable and which are unstable?