UNIVERSITY OF MISSISSIPPI

Department of Physics and Astronomy

Advanced Mechanics I (Phys. 709) — Prof. Leo C. Stein — Fall 2018

Problem Set 6 — Midterm

Material: The midterm covers the material so far except for last week and this week (linear oscillators).

Due date: Tuesday, Oct. 23, 2018 by 5PM to 205 Lewis Hall. If my door is closed, please slide the exam under my door. Late exams will require extenuating circumstances.

Logistics: The exam consists of this page plus two pages of questions. Do not look at the problems until you are ready to start it.

Time: The work might expand to eat up as much time as you allot – therefore I highly recommend you restrict yourself to no more than 18 hours cumulative time spent on these problems. You may take as many breaks as you like, not counted against the 18 hours. You should not be consulting references, working on the problems, or discussing with others during the breaks.

Resources: The midterm and final are not collaborative. All questions must be done on your own, without consulting anyone else. You may consult your own notes (both in-class and notes on this class you or a colleague in the class have made), the textbook by Fetter and Walecka (or Hand and Finch if you have it), and solution sets on the course website. You may not consult any other material, including other textbooks, the web (except for the current Phys. 709 website), material from previous years' Phys. 709 or any other classes, or copies you have made of such material, or any other resources. Calculators and symbolic manipulation programs are not allowed.

1. **A few systems**.

   (a) **Spring pendulum**. A pendulum is made from a mass m hanging in gravity g on a spring with the other end fixed. The undistorted length of the spring is l and the spring constant is k. Find the Lagrangian for the full three dimensional motion using spherical polar coordinates r, θ, ϕ with the origin at the fixed end of the spring. What are the constants of the motion, if any?

   (b) **Bead on a wire**. A bead of mass m slides in gravity g on a frictionless wire in a vertical plane with shape described by the equation $x^3 + xz + z^3 = 1$ (with x horizontal, z vertical). Find the complete set of equations that could be analyzed (e.g. put on a computer) to give the motion.

   (c) **Finding a Hamiltonian**. The Lagrangian of a one-degree of freedom system $L = T - V$ in the generalized coordinate q is given by

   $$T = \tfrac{1}{2} m\dot{q}^2 + \tfrac{1}{2} m\omega^2 \sin^2 \alpha\, q^2, \quad V = mg \cos\alpha\, q, \tag{1}$$

   with m, g, α, ω constants. What is the Hamiltonian as a function of coordinate and momentum (not velocity)?

   (d) Consider the Lagrangian

   $$L = \dot{q}_1^2 + \frac{\dot{q}_2^2}{a + bq_1^2} + k_1 q_1^2 + k_2 \dot{q}_1 \dot{q}_2 \,, \tag{2}$$

   where a, b, k_1 and k_2 are constants. Find the corresponding Hamiltonian, and write out the four Hamilton's equations.

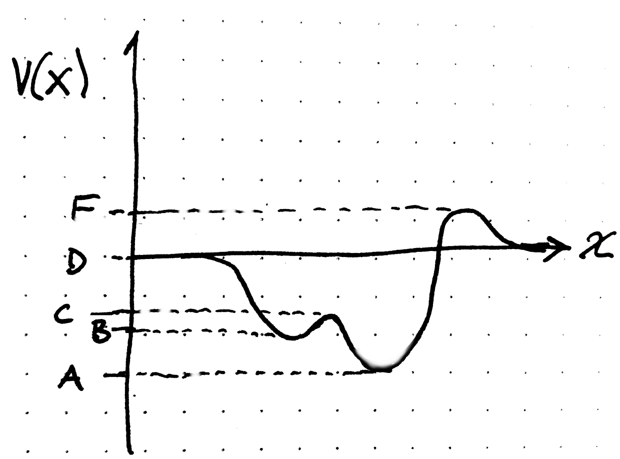
2. **Rolling cylinder on ramp:** A hollow cylinder of mass M and radius a is rolling down a ramp of angle α with axis horizontal (i.e. you may consider the problem as two dimensional, equivalent to a hoop rolling on a line) in gravity g. Use as coordinates d, the distance of the contact line from the top of the ramp, and ϕ, the angle of a fixed radius on the circular cross-section to the vertical.

   (a) What is the Lagrangian?

   (b) Is the rolling constraint holonomic or nonholonomic in this case?

   (c) Use the Lagrangian approach with a Lagrange multiplier to implement the rolling constraint to find the equations of rolling motion, i.e. explicit equations for $\ddot{d}$ and $\ddot{\phi}$ in terms of m, a, g, α, and hence find $d(t), \phi(t)$ for an initial condition at time $t = 0$ of a stationary cylinder at $d = 0, \phi = 0$.

3. **Potential analysis**. Consider the following potential $V(x)$. The potential takes on the value $V(x) = D$ outside of the region shown.



Suppose there is a one-degree of freedom system where the energy is given by

$$E = \frac{1}{2}(\dot{x})^2 + V(x)\,. \tag{3}$$

(a) On your own sketch of this potential, mark all the x locations where equilibrium is possible. For each value of x, report whether the equilibrium is stable or unstable.

(b) For what values of energy E is motion possible? (This is in relation to the potential values A, B, C, D, and F).

(c) Choose a value of energy $B < E < C$. Denote the allowed ranges of x for that given energy, on a new sketch of the potential.

(d) Choose a value of energy $C < E < D$. Denote the allowed ranges of x for that given energy, on a new sketch of the potential.

(e) Choose a value of energy $D < E < F$. Denote the allowed ranges of x for that given energy, on a new sketch of the potential.

(f) How many qualitatively distinct types of motion are possible for particles that asymptote to either $x \to \pm\infty$ as time goes to $t \to \pm\infty$ (in any combination of endpoints)? For each of these types of motion: what value(s) of energy E is required for that type of trajectory?

4. **Canonical transformations**.

(a) Suppose we start on a phase space with canonical coordinates (q_1, p_1, q_2, p_2). Show that the following transformation is canonical:

$$Q_1 = q_1^2 \,, \qquad Q_2 = q_2 \sec p_2 \,, \tag{4}$$

$$P_1 = \frac{p_1 \cos p_2 - 2q_2}{2q_1 \cos p_2} \,, \qquad P_2 = \sin p_2 - 2q_1 \,. \tag{5}$$

(b) Determine whether or not the following transformation is canonical:

$$Q_1 = q_1 q_2 \,, \qquad P_1 = \frac{p_1 - p_2}{q_2 - q_1} + 1 \,, \tag{6}$$

$$Q_2 = q_1 + q_2 \,, \qquad P_2 = \frac{q_2 p_2 - q_1 p_1}{q_2 - q_1} - (q_2 + q_1) \,. \tag{7}$$

(c) Now we go to a 2-dimensional phase space with canonical coordinates (q, p). Find the values of α and β such that the following is a canonical transformation:

$$Q = q^\alpha \cos \beta p \,, \qquad P = q^\alpha \sin \beta p \,. \tag{8}$$

5. **Some Poisson brackets**. Consider a 6-dimensional phase space (x, p_x, y, p_y, z, p_z). Let $a = xp_y - yp_x$, and $b = zp_x - xp_z$. Compute the following Poisson brackets: $\{a, x\}$, $\{a, y\}$, $\{a, z\}$, $\{a, p_x\}$, $\{a, p_y\}$, $\{a, p_z\}$, $\{b, x\}$, $\{b, y\}$, $\{b, z\}$, and $\{a, b\}$. Explain in words what is the "action" of a and b on phase space; what do they represent?