UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Advanced Mechanics I (Phys. 709) — Prof. Leo C. Stein — Fall 2018

Problem Set 9 — Final

Material: The final covers the material so far except for this week (Lyapunov exponents and parametric resonance).

Due date: Thursday, Dec. 6, 2018 by 5PM to 205 Lewis Hall. If my door is closed, please slide the exam under my door. Late exams will require extenuating circumstances.

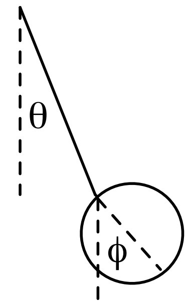
Logistics: The exam consists of this page plus three pages of questions. Do not look at the problems until you are ready to start it.

Time: The work might expand to eat up as much time as you allot – therefore I highly recommend you restrict yourself to no more than two days spent on these problems. You may take as many breaks as you like. You should not be consulting references, working on the problems, or discussing with others during the breaks.

Resources: The midterm and final are not collaborative. All questions must be done on your own, without consulting anyone else. You may consult your own notes (both in-class and notes on this class you or a colleague in the class have made), the textbook by Fetter and Walecka (or Hand and Finch, or Goldstein if you have it), and solution sets on the course website. You may not consult any other material, including other textbooks, the web (except for the current Phys. 709 website), material from previous years' Phys. 709 or any other classes, or copies you have made of such material, or any other resources. Calculators and symbolic manipulation programs are not allowed.

1. **A variety of topics**

(a) A pendulum consists of a circular disc of radius R and mass M freely suspended from a point on its circumference attached to the bottom end of a rod of negligible mass and length l. The rod is free to swing about its top end. Take the acceleration due to gravity to be g. Use as variables: θ, the angle of the rod to the vertical; and ϕ the angle of the disc diameter through its suspension point to the vertical.
Find the Lagrangian that describes the motion in which the rod and disc remain in the $x-z$ plane.



(b) A vector with components $(2, 1, 2)$ with respect to Cartesian axes is rotated through $60°$ about the x-axis. What are the components of the rotated vector?

(c) i. Sketch a one-dimensional potential which has no equilibrium points (and give some asymptotic behavior for $V(r)$ at large $|r|$).

ii. Sketch a one-dimensional potential which has one unstable equilibrium point (and give some asymptotic behavior for $V(r)$ at large $|r|$).

iii. Write down an asymptotic form for $V(r)$ at large $|r|$ such that particles with energy $E \geq 1$ can escape to spatial infinity, but particles with $E < 1$ will remain in a bound orbit.

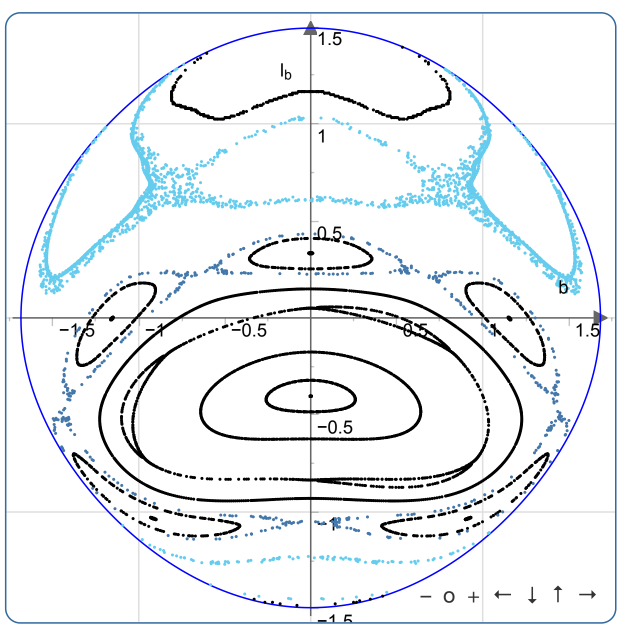
(d) Find the explicit expressions $Q(q,p), P(q,p)$ for the canonical transformation given by the type 1 generating function

$$F_1(q,Q) = q\sin^{-1}(qe^Q) + \sqrt{e^{-2Q} - q^2}. \tag{1}$$

(e) After finding the explicit transformation in the previous step, perform the Legendre transform to a type 2 generating function $F_2 = F_1 + QP$. Make sure to note which two variables are independent, and that only those two appear in the explicit expression for F_2.

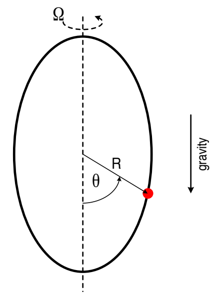
(f) In words, what does it mean for a system to be integrable vs. for a system to be chaotic?

(g) Refer to the Poincaré section below. Locate examples of the following features on the diagram: i. Quasiperiodic orbits that are not fixpoints; ii. hyperbolic fixpoints; iii. elliptic (stable) fixpoints; iv. an orbit in a chaotic region of phase space



2. **Bead on rotating hoop**.

A bead of mass m moves without friction on a circular hoop of radius R which rotates at a fixed angular frequency Ω about the vertical z axis in gravity g, as shown in the figure.



Answer the following:

(a) Pick out the correct word(s) describing the constraint (that the wire remain on the bead) from the following list: holonomic, nonholonomic, time dependent (rheonomic), time independent (scleronomic).

(b) The constrained dynamics has one degree of freedom, that may be described by the generalized coordinate θ. Find the generalized force $\mathcal{F}_\theta$ when the mass is at angle θ.

(c) Derive expressions for the kinetic energy T and potential energy V, and show that these lead to the expression for the Lagrangian L (up to constants etc.)

$$L = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m R^2 \sin^2\theta \, \Omega^2 + m g R \cos\theta \tag{2}$$

Use this expression for the Lagrangian to answer the following:

(d) Find the equation of motion.

(e) What is the Hamiltonian as a function of $\theta, \dot{\theta}$, and is it a constant of the motion?

(f) Is the total energy $T + V$ a constant of the motion? If not, where does the energy come from to change the total energy (potential + kinetic) of the bead?

(g) Show that $\theta = 0$ and $\theta = \pi$ are equilibria (θ is time independent) for all rotation rates. Are the equilibria stable or unstable? Does this change as a function of Ω?

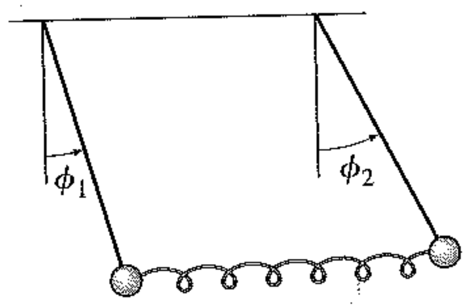
(h) Show that for rotation rates Ω greater than a critical rotation rate Ω_c an additional pair of equilibria develop at $\pm\theta_r$ and find θ_r as a function of g, R, and Ω.

(i) Is the new solution pair stable? If so, find the frequency of linear oscillations about them.

Finally:

(j) Write down an effective action that would allow you to solve the problem using Cartesian coordinates x, y, z and the method of Lagrange multipliers.

3. **Coupled pendulums:** Two pendulums of identical length l with masses m and $2m$ are connected by a spring of spring constant $k = 2mg/l$. Use the variables ϕ_1 for the lighter mass and ϕ_2 for the heavier mass. The spring is unstretched when the two pendulums are vertical $\phi_1 = \phi_2 = 0$.



   (a) What is the Lagrangian up to quadratic order in the small displacement and velocities?

   (b) Find the 2×2 kinetic and potential energy matrices.

   (c) Using these, show that the normal mode frequencies are ω_0 and $2\omega_0$ with $\omega_0 = \sqrt{g/l}$.

   (d) For an initial condition of both pendulum released from rest at $t = 0$, find the ratio of the displacements $\phi_1(0)/\phi_2(0)$ such that only the higher frequency oscillation is excited.

4. **Action-angle variables**. A particle of mass m with one-dimensional coordinate q is moving in a potential $V(q) = aq^4$ with $a > 0$.

   (a) Find the Hamiltonian $H(q, p)$ and the equations of motion for q, p.

   (b) Sketch a few representative trajectories of solutions in phase space.

   (c) Show that the action variable I as a function of the energy E for an oscillating solution is given by

$$I = \alpha \frac{E^{3/4}}{a^{1/4}}, \tag{3}$$

   and find the constant α in terms of the mass m and the constant c defined by $c = \int_0^1 \sqrt{1 - x^4}\, dx$ (its value is about 0.8740).

   (d) Find the frequency of the oscillations at energy E.

   (e) If the parameter a is *slowly* varied, how does the amplitude of oscillation (i.e. the maximum value of q) vary with a?