UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism I (Phys. 401) — Prof. Leo C. Stein — Fall 2019

Problem Set 1 — SOLUTIONS

Due: Thursday, Sept. 5, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. Prove the following vector algebra and calculus identities. By prove I mean to show the list of steps with enough detail and justification (e.g. stating "because of antisymmetry of the cross product") so that somebody just learning this topic could follow the derivations, and be convinced of their correctness. Breaking things up into components is a perfectly valid strategy. Boldface symbols are vectors.

$$\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v}(\boldsymbol{u} \cdot \boldsymbol{w}) - \boldsymbol{w}(\boldsymbol{u} \cdot \boldsymbol{v})\,, \tag{1}$$
$$0 = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b})\,, \tag{2}$$
$$\nabla(fg) = (\nabla f)g + f\nabla g\,, \tag{3}$$
$$\nabla \cdot (f\boldsymbol{v}) = (\nabla f) \cdot \boldsymbol{v} + f\nabla \cdot \boldsymbol{v}\,, \tag{4}$$
$$\nabla \cdot (\nabla \times \boldsymbol{v}) = 0\,. \tag{5}$$

Solution: All of these can be proved by expanding both sides in components, and using the definitions of the dot/cross products, divergence/grad/curl, and the product rule for partial derivatives of functions. I will give proofs here for identities which can be proved more easily using "index gymnastics" (tensor calculus in index notation).

In indices, the dot/cross product and grad/div/curl are

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i \qquad (\boldsymbol{a} \times \boldsymbol{b})^i = \epsilon^{ijk} a_j b_k \tag{6}$$
$$(\nabla f)_i = \partial_i f \qquad \nabla \cdot \boldsymbol{a} = \partial_i a^i \qquad (\nabla \cdot \boldsymbol{a})^i = \epsilon^{ijk} \partial_j a_k \tag{7}$$

where $\partial_i = \partial/\partial x^i$ and recall that $x^i = (x, y, z)$ for $i = 1, 2, 3$ respectively. Above we are using the Einstein summation convention (repeated indices are summed) and ϵ^{ijk} is the completely anti-symmetric Levi-Civita tensor, meaning $\epsilon^{jik} = -\epsilon^{ijk}$, antisymmetric on every pair of indices.

To prove Eq. (1) requires the identity

$$\epsilon^{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta^i_l & \delta^j_l & \delta^k_l \\ \delta^i_m & \delta^j_m & \delta^k_m \\ \delta^i_n & \delta^j_n & \delta^k_n \end{vmatrix} \tag{8}$$

which is easy to generalize to any dimension. The expression in Eq. (1) is written in index notation as

$$\epsilon^{ijk} u^j \epsilon^{klm} v^l w^m = (\delta^i_l \delta^j_m - \delta^i_m \delta^j_l) u^j v^l w^m\,, \tag{9}$$

after the contraction between the two epsilons has been performed. Then performing the remaining contractions with the deltas gives the dot product expression desired.

Eqs. (3) and (4) are proved by the fact that ∇_i acts as a derivation acting on any type of tensors, e.g.

$$\nabla_i(fg) = (\nabla_i f)g + f(\nabla_i g)\,, \qquad \nabla_i(fv^j) = (\nabla_i f)v^j + f(\nabla_i v^j)\,. \tag{10}$$

Eq. (4) results from contracting the latter equality above with δ^i_j.

Finally, Eq. (5) is proved easily by using the symmetries of the Levi-Civita tensor and the fact that partial derivatives commute. In indices that equation is

$$\nabla_i \epsilon^{ijk} \nabla_j v_k = -\nabla_i \epsilon^{jik} \nabla_j v_k = -\nabla_j \epsilon^{jik} \nabla_i v_k = -\nabla_i \epsilon^{ijk} \nabla_j v_k = 0 \,. \tag{11}$$

The first equality is by antisymmetry of Levi-Civita. The second equality is by symmetry of partial derivatives. The third equality is by exchanging "dummy" indices $i \leftrightarrows j$. Now we have shown that a quantity is the negative of itself, and therefore equals zero. This general approach works whenever you have a pair of anti-symmetric indices contracted with a pair of symmetric indices.

2. Let's define the function

$$p(x,y,z) = ax + b^2 y^2 - c^2 z^2 \,, \tag{12}$$

where a, b, c are nonzero real numbers. The set of points with coordinates (x, y, z) that evaluate to $p(x, y, z) = 0$ make a "hyperbolic paraboloid" surface.

(a) What is the gradient ∇p of this function?
Solution:

$$\nabla p = a\hat{\boldsymbol{x}} + 2b^2 y \hat{\boldsymbol{y}} - 2c^2 z \hat{\boldsymbol{z}} \,. \tag{13}$$

(b) Find the *unit* normal vector $\hat{\boldsymbol{n}}$ to the surface $p(x, y, z) = 0$.
Solution: The gradient is normal to a level set of some function, as discussed in class. All we have to do is normalize the above gradient, so first take its norm squared:

$$|\nabla p|^2 = a^2 + 4b^4 y^2 + 4c^4 z^2 \,. \tag{14}$$

Now we have the unit normal vector:

$$\hat{\boldsymbol{n}} = \frac{a\hat{\boldsymbol{x}} + 2b^2 y \hat{\boldsymbol{y}} - 2c^2 z \hat{\boldsymbol{z}}}{\sqrt{a^2 + 4b^4 y^2 + 4c^4 z^2}} \,. \tag{15}$$

(c) Evaluate the unit normal at the point $(1/a, 2/b, \sqrt{5}/c)$.
Solution: We just insert the coordinates into the above unit vector, finding

$$\hat{\boldsymbol{n}}(1/a, 2/b, \sqrt{5}/c) = \frac{a\hat{\boldsymbol{x}} + 4b\hat{\boldsymbol{y}} - 2\sqrt{5}c\hat{\boldsymbol{z}}}{\sqrt{a^2 + 16b^2 + 20c^2}} \,. \tag{16}$$

3. Recall our definition of the vector field $\boldsymbol{r} \equiv x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$. Let's also define a constant vector $\boldsymbol{k}$ with components (k_x, k_y, k_z). Compute the following quantities:

(a) $\nabla \cdot \boldsymbol{r}$
Solution: $\nabla \cdot \boldsymbol{r} = \sum_{i=1}^3 \partial_i x^i = \sum_{i=1}^3 1 = 3$.

(b) $\nabla \times \boldsymbol{r}$
Solution: $(\nabla \times \boldsymbol{r})^i = \epsilon^{ijk} \nabla_j r_k = \epsilon^{ijk} \delta_{jk} = 0$

(c) $\nabla \cdot \hat{\boldsymbol{r}}$
Solution: The hatted (unit) vector is $(\hat{\boldsymbol{r}})^i = x^i/r$ where $r = \sqrt{\boldsymbol{r} \cdot \boldsymbol{r}} = \sqrt{x^2 + y^2 + z^2}$. Taking the divergence we get

$$\nabla_i \hat{r}^i = \partial_i \frac{x^i}{r} = \frac{\partial_i x^i}{r} + x^i \partial_i \frac{1}{r} = \frac{3}{r} + x^i \frac{-1}{r^2} \partial_i r \tag{17}$$

Now we need to know the gradient of r, $(\nabla r)_i = \partial_i r$.

$$\nabla r = \frac{1}{2r}(2x\hat{\boldsymbol{x}} + 2y\hat{\boldsymbol{y}} + 2z\hat{\boldsymbol{z}}) = \frac{\boldsymbol{r}}{r} = \hat{\boldsymbol{r}} \,. \tag{18}$$

Putting it together we find

$$\nabla \cdot \hat{\boldsymbol{r}} = \frac{3}{r} - \frac{x^i x^i}{r^3} = \frac{3}{r} - \frac{r^2}{r^3} = \frac{2}{r} \,. \tag{19}$$

(d) $\nabla \times \hat{\boldsymbol{r}}$

Solution: $\epsilon^{ijk}\nabla_j \frac{r^k}{r} = \epsilon^{ijk}(\frac{\delta_{jk}}{r} + r^k \nabla_j \frac{1}{r}) = 0 + \epsilon^{ijk} r^j \frac{-1}{r^2}\frac{r^k}{r} = 0$

(e) $\nabla \times (\boldsymbol{k} \times \boldsymbol{r})$

Solution: Since $\boldsymbol{k}$ is a constant (position-independent) vector, its components go through derivatives. This is easiest done in terms of index notation,

$$\epsilon^{ijl}\nabla_j \epsilon_{lmn} k^m r^n = (\delta^i_m \delta^j_n - \delta^i_n \delta^j_m) k^m \nabla_j r^n = (\delta^i_m \delta^j_n - \delta^i_n \delta^j_m) k^m \delta^n_j = 3k^i - k^i = 2k^i \, . \tag{20}$$

(f) $\nabla(\boldsymbol{k} \cdot \boldsymbol{r})$

Solution: $\nabla_i (k^j x^j) = k^j \delta^j_i = k_i$.