UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism I (Phys. 401) — Prof. Leo C. Stein — Fall 2019

Problem Set 2 — SOLUTIONS

Due: Thursday, Sept. 12, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. Let's define the vector field $\boldsymbol{V} = xy\hat{\boldsymbol{x}} - \frac{3}{2}y^2\hat{\boldsymbol{y}}$. Evaluate the line integral

$$I = \int_{\mathcal{P}} \boldsymbol{V} \cdot d\boldsymbol{l}\,, \tag{1}$$

where the path $\mathcal{P}$ starts at $(x, y) = (0, 0)$ and goes to $(1, 2)$ along the path parameterized by $\boldsymbol{\gamma}(\lambda) = (\lambda, 2\lambda^3)$.

Solution: The starting value of λ is 0 since $\boldsymbol{\gamma}(0) = (0, 0)$, and the ending value is 1 since $\boldsymbol{\gamma}(1) = (1, 2)$. The line element is

$$d\boldsymbol{l} = \boldsymbol{\gamma}'(\lambda)d\lambda = (1, 6\lambda^2)d\lambda\,. \tag{2}$$

Evaluating $\boldsymbol{V}$ along the trajectory $\boldsymbol{\gamma}$,

$$\boldsymbol{V}(\boldsymbol{\gamma}(\lambda)) = (2\lambda^4, -\frac{3}{2}4\lambda^6)\,. \tag{3}$$

Taking the dot product, now we have an ordinary one-dimensional integral we can do,

$$I = \int_0^1 (2\lambda^4 - 36\lambda^8)d\lambda = -\frac{18}{5}\,. \tag{4}$$

2. **Checking the divergence theorem**. Consider the vector field

$$\boldsymbol{A} = xy^2\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + xyz\hat{\boldsymbol{z}}\,. \tag{5}$$

Consider a rectangular prism, aligned with the xyz axes, of length 1 in the x direction, width 1 in the y direction, and some arbitrary height h in the z direction. Let one corner of this rectangular prism sit at $(0, 0, 0)$ and the opposite corner be at $(1, 1, h)$. We will call the interior $\mathcal{V}$ and the surface $\partial\mathcal{V} = \mathcal{S}$. Recall that a surface has an orientation; we take the orientation of the 6 rectangular faces to be pointing out.

(a) Compute the six surface integrals over the 6 oriented faces and add them up to find the total surface integral

$$I = \oint_{\mathcal{S}} \boldsymbol{A} \cdot d\boldsymbol{a}\,. \tag{6}$$

Solution: There are 6 faces to work on with integrals $I_1, \ldots I_6$.

i. I_1, the "bottom" face $z = 0, 0 \le x \le 1, 0 \le y \le 1$. The outward area element is $d\boldsymbol{a} = -\hat{\boldsymbol{z}}dxdy$. The integrand is the dot product $\boldsymbol{A} \cdot d\boldsymbol{a} = -xyzdxdy$. Since $z = 0$ on this face, the integrand vanishes, $I_1 = 0$.

ii. I_2, the "top" face $z = h, 0 \le x \le 1, 0 \le y \le 1$. The outward area element is $d\boldsymbol{a} = +\hat{\boldsymbol{z}}dxdy$. The integrand is the dot product $\boldsymbol{A} \cdot d\boldsymbol{a} = +xyzdxdy$. Here $z = h$, so we get the integral

$$I_2 = h \int_0^1 dx \int_0^1 dy\, xy = \frac{h}{4}\,. \tag{7}$$

iii. I_3, the "left" face $x = 0, 0 \le y \le 1, 0 \le z \le h$. The outward area element is $d\boldsymbol{a} = -\hat{\boldsymbol{x}}dydz$. The integrand is the dot product $\boldsymbol{A} \cdot d\boldsymbol{a} = -xy^2dydz$. Here $x = 0$ so the integrand vanishes, $I_3 = 0$.

iv. I_4, the "right" face $x = 1, 0 \le y \le 1, 0 \le z \le h$. The outward area element is $d\boldsymbol{a} = +\hat{\boldsymbol{x}}dydz$. The integrand is the dot product $\boldsymbol{A} \cdot d\boldsymbol{a} = +xy^2dydz$. Here $x = 1$, so we get the integral

$$I_4 = \int_0^1 dy \int_0^h dzy^2 = \frac{h}{3}\,. \tag{8}$$

v. I_5, the "front" face $y = 0, 0 \le x \le 1, 0 \le z \le h$. The outward area element is $d\boldsymbol{a} = -\hat{\boldsymbol{y}}dxdz$. The integrand is the dot product $\boldsymbol{A} \cdot d\boldsymbol{a} = -ydxdz$. Here $y = 0$ so the integrand vanishes, $I_5 = 0$.

vi. I_6, the "back" face $y = 1, 0 \le x \le 1, 0 \le z \le h$. The outward area element is $d\boldsymbol{a} == \hat{\boldsymbol{y}}dxdz$. The integrand is the dot product $\boldsymbol{A} \cdot d\boldsymbol{a} = +ydxdz$. Here $y = 1$, so we get the integral

$$I_6 = \int_0^1 dx \int_0^h dz1 = h\,. \tag{9}$$

Adding these up we get the total integral

$$I = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = 0 + \frac{h}{4} + 0 + \frac{h}{3} + 0 + h = \frac{19h}{12}\,. \tag{10}$$

(b) Use the divergence theorem to turn the above integral into a volume integral, and evaluate the volume integral to verify that the two approaches give the same result.

Solution: From the divergence theorem,

$$I = \int_{\mathcal{V}} (\nabla \cdot \boldsymbol{A}) d^3\text{Vol}\,. \tag{11}$$

The divergence is given by

$$\nabla \cdot \boldsymbol{A} = \frac{\partial xy^2}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial xyz}{\partial z} = y^2 + 1 + xy\,. \tag{12}$$

Now the ordinary iterated integral to do is

$$I = \int_0^1 dx \int_0^1 dy \int_0^h dz(y^2 + 1 + xy) = \frac{19h}{12}\,. \tag{13}$$

This agrees with the surface integral we found earlier.

3. **Various surfaces**.

(a) Let's define a parametric 2-dimensional surface in 3-dimensional space with the functions

$$\boldsymbol{\sigma}(u, v) \equiv (v \cos u, v \sin u, v)\,, \tag{14}$$

where u, v are two parameters along the surface. Find the oriented differential area element $d\boldsymbol{a}$ at some point with parameters (u, v). Describe in words the shape of this surface.

Solution: For a parametric 2-dimensional surface $\boldsymbol{\sigma}(u, v)$, the oriented differential area element is

$$d\boldsymbol{a} = \frac{\partial \boldsymbol{\sigma}}{\partial u} \times \frac{\partial \boldsymbol{\sigma}}{\partial v} du dv \,. \tag{15}$$

For the given surface, these (tangent) vectors are

$$\frac{\partial \boldsymbol{\sigma}}{\partial u} = (-v \sin u, v \cos u, 0) \,, \tag{16}$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial u} = (\cos u, \sin u, 1) \,. \tag{17}$$

Taking the cross product we get

$$\frac{\partial \boldsymbol{\sigma}}{\partial u} \times \frac{\partial \boldsymbol{\sigma}}{\partial v} = (v \cos u, v \sin u, -v \cos^2 u - v \sin^2 u) = (v \cos u, v \sin u, -1) \,. \tag{18}$$

Description of surface: If we fix a value of v, and let u vary, this parameterization will trace out a circle that lies in the plane $z = v$. The radius of that circle is v. Therefore, each plane of constant z contains a circle whose radius is the same as the height of the plane. This stack of circles is a cone with opening angle 90°.

(b) Besides a parametric definition (like above) and an implicit definition (like a set of points satisfying some equation $f(x, y, z) = 0$), we can also define a surface via the *graph* of some function $z = g(x, y)$.

i. What is a function $f(x, y, z)$ such that the points $f(x, y, z) = 0$ are the same as the set of points on the graph $z = g(x, y)$? Find a normal to surface $f(x, y, z) = 0$ in terms of $g(x, y)$.
Solution: Simply rearrange the graph's equality so it reads as (something that depends on x, y, z) = (some constant). This is as simple as subtracting $g(x, y)$ from both sides. So we go from the graph $z = g(x, y)$ immediately to

$$f(x, y, z) = z - g(x, y) = 0 \,. \tag{19}$$

The normal to the graph is therefore the gradient of this new function f,

$$\boldsymbol{n} = \nabla f = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \right) \,. \tag{20}$$

ii. Give a parametric form of the same surface, i.e. find a $\boldsymbol{\sigma}(u, v)$ which returns the same set of points as the graph $z = g(x, y)$. Find the oriented differential area element $d\boldsymbol{a}$ to this parametric surface. Check that this area element points in the same direction as the normal you found above.
Solution: When we have a graph, the x and y arguments are already parameters of the height. So, we just have to add in the x and y values that are implied,

$$\boldsymbol{\sigma}(u, v) = (u, v, g(u, v)) \,. \tag{21}$$

Now let's compute

$$d\boldsymbol{a} = \frac{\partial \boldsymbol{\sigma}}{\partial u} \times \frac{\partial \boldsymbol{\sigma}}{\partial v} du dv \,. \tag{22}$$

We have

$$\frac{\partial \boldsymbol{\sigma}}{\partial u} = \left(1, 0, \frac{\partial g}{\partial u} \right) \,, \tag{23}$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial v} = \left(0, 1, \frac{\partial g}{\partial v} \right) \,. \tag{24}$$

Taking the cross product we get

$$\frac{\partial \boldsymbol{\sigma}}{\partial u} \times \frac{\partial \boldsymbol{\sigma}}{\partial v} = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \right) \,, \tag{25}$$

the same as $\boldsymbol{n} = \nabla f$ we found above.