UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism I (Phys. 401) — Prof. Leo C. Stein — Fall 2019

Problem Set 3 — SOLUTIONS

Due: Friday, Sept. 20, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. **Some "integration by parts" formulas**

   (a) The following is an identity for differentiable functions f and differentiable vector fields $\boldsymbol{V}$:

   $$\int_{\mathcal{S}} f(\nabla\times\boldsymbol{V})\cdot d\boldsymbol{a} = \int_{\mathcal{S}}(\boldsymbol{V}\times\nabla f)\cdot d\boldsymbol{a} + \oint_{\partial\mathcal{S}} f\boldsymbol{V}\cdot d\boldsymbol{l}\,. \tag{1}$$

   Which "Leibniz rule" (product rule) leads to this integral identity? Starting with that rule, and one of the fundamental theorems, prove the above identity.

   Solution: We start from the product rule

   $$\nabla\times(f\boldsymbol{V}) = f(\nabla\times\boldsymbol{V}) - \boldsymbol{V}\times\nabla f\,. \tag{2}$$

   Solve for $f(\nabla\times\boldsymbol{V})$, then integrate both sides of the equality over some surface $\mathcal{S}$,

   $$\int_{\mathcal{S}} f(\nabla\times\boldsymbol{V})\cdot d\boldsymbol{a} = \int_{\mathcal{S}}(\boldsymbol{V}\times\nabla f)\cdot d\boldsymbol{a} + \int_{\mathcal{S}}[\nabla\times(f\boldsymbol{V})]\cdot d\boldsymbol{a}\,. \tag{3}$$

   The last term can be converted to a line integral around the closed loop $\partial\mathcal{S}$ via the curl theorem, giving the desired identity.

   (b) The following is an identity for two differentiable vector fields $\boldsymbol{V},\boldsymbol{W}$:

   $$\int_{\mathcal{V}} \boldsymbol{V}\cdot(\nabla\times\boldsymbol{W})\,d^3\mathrm{Vol} = \int_{\mathcal{V}} \boldsymbol{W}\cdot(\nabla\times\boldsymbol{V})\,d^3\mathrm{Vol} + \oint_{\partial\mathcal{V}}(\boldsymbol{W}\times\boldsymbol{V})\cdot d\boldsymbol{a}\,. \tag{4}$$

   Which "Leibniz rule" (product rule) leads to this integral identity? Starting with that rule, and one of the fundamental theorems, prove the above identity.

   Solution: The product rule to start from is the divergence of a cross product of vectors,

   $$\nabla\cdot(\boldsymbol{V}\times\boldsymbol{W}) = \boldsymbol{W}\cdot(\nabla\times\boldsymbol{V}) - \boldsymbol{V}\cdot(\nabla\times\boldsymbol{W})\,. \tag{5}$$

   Now solve for $\boldsymbol{V}\cdot(\nabla\times\boldsymbol{W})$ and integrate both sides of the equality over some volume $\mathcal{V}$,

   $$\int_{\mathcal{V}} \boldsymbol{V}\cdot(\nabla\times\boldsymbol{W})\,d^3\mathrm{Vol} = \int_{\mathcal{V}} \boldsymbol{W}\cdot(\nabla\times\boldsymbol{V})\,d^3\mathrm{Vol} - \int_{\mathcal{V}}\nabla\cdot(\boldsymbol{V}\times\boldsymbol{W})\,d^3\mathrm{Vol}\,. \tag{6}$$

   The last term can be converted to a surface integral over the closed surface $\partial\mathcal{V}$ by using the divergence theorem. Finally reverse the order of $\boldsymbol{V}$ and $\boldsymbol{W}$ in the last term to get rid of the minus sign, giving the desired identity.

2. (a) Recall that $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$, where $\boldsymbol{r}'$ is some other fixed point. Show the identity

   $$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,, \tag{7}$$

where the gradient ∇ just takes derivatives with respect to $\boldsymbol{r} = (x, y, z)$.

Solution: It is simplest to write $1/\mathscr{r} = (\boldsymbol{\mathscr{r}} \cdot \boldsymbol{\mathscr{r}})^{-1/2}$. Then using the chain rule,

$$\nabla_i \left(\mathscr{r}^{-1}\right) = \nabla \left((\boldsymbol{\mathscr{r}}^j \boldsymbol{\mathscr{r}}^j)^{-1/2}\right) = -\frac{1}{2}(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{\mathscr{r}})^{-3/2} 2\boldsymbol{\mathscr{r}}^j \nabla_i \boldsymbol{\mathscr{r}}^j \,. \tag{8}$$

Now let's compute

$$\nabla_i \boldsymbol{\mathscr{r}}^j = \nabla_i (\boldsymbol{r}^j - \boldsymbol{r}'^j) = \delta_i^j \,, \tag{9}$$

because $\boldsymbol{r}'$ is a constant with respect to $\boldsymbol{r}$. So, we found

$$\nabla_i \left(\frac{1}{\mathscr{r}}\right) = \frac{-1}{\mathscr{r}^3} \boldsymbol{\mathscr{r}}^j \delta_i^j = \frac{-1}{\mathscr{r}^3} \boldsymbol{\mathscr{r}}^i = \frac{-1}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}^i \,. \tag{10}$$

(b) Find the next derivative – now we have to use index notation. We want the whole tensor of second derivatives,

$$\nabla_i \nabla_j \left(\frac{1}{\mathscr{r}}\right) \,, \tag{11}$$

which you can think of as a 3x3 matrix. But, you can find a much more compact way to write it in terms of $\boldsymbol{\mathscr{r}}_k$.

Solution: Now we use the product rule and again the chain rule. First plugging in the previous result, then using the product rule,

$$\nabla_i \nabla_j \left(\frac{1}{\mathscr{r}}\right) = \nabla_i \frac{-\boldsymbol{\mathscr{r}}^j}{\mathscr{r}^3} = \frac{-\nabla_i \boldsymbol{\mathscr{r}}^j}{\mathscr{r}^3} + (-\boldsymbol{\mathscr{r}}^j) \nabla_i \left(\frac{1}{\mathscr{r}^3}\right) \,. \tag{12}$$

As above, $\nabla_i \boldsymbol{\mathscr{r}}^j = \delta_i^j$. We proceed with $\nabla_i \mathscr{r}^{-3}$ as before.

$$\nabla_i \nabla_j \left(\frac{1}{\mathscr{r}}\right) = \frac{-\delta_i^j}{\mathscr{r}^3} + (-\boldsymbol{\mathscr{r}}^j) \left(\frac{-3}{2\mathscr{r}^5} 2\boldsymbol{\mathscr{r}}^k \nabla_i \boldsymbol{\mathscr{r}}^k\right) = -\frac{\delta_i^j \mathscr{r}^2 + 3\boldsymbol{\mathscr{r}}^i \boldsymbol{\mathscr{r}}^j}{\mathscr{r}^5} \,. \tag{13}$$

A good check here is that if you take the trace over i, j, you get the Laplacian of $1/\mathscr{r}$, which vanishes away from the origin. So to be absolutely correct, we should also add in a delta function to the above, but I have omitted this.

3. **When a vector field is conservative...** Consider the vector field

$$\boldsymbol{F} = \frac{\sin z}{x + y^2} \hat{\boldsymbol{x}} + \frac{2y \sin z}{x + y^2} \hat{\boldsymbol{y}} + \cos(z) \ln(x + y^2) \hat{\boldsymbol{z}} \,. \tag{14}$$

(a) Show that this vector field is conservative.

Solution: To show that the vector field is conservative, take its curl. We find that $\nabla \times \boldsymbol{F} = 0$, thus $\boldsymbol{F}$ is conservative.

(b) Find a scalar potential (up to an additive constant) such that $\boldsymbol{F} = -\nabla V$.

Solution: Since the vector field is conservative, we can integrate from any initial point $\boldsymbol{a}$ to $\boldsymbol{r}$ to get the potential $V(\boldsymbol{r})$,

$$V(\boldsymbol{r}) = V(\boldsymbol{a}) + \int_{\boldsymbol{a}}^{\boldsymbol{r}} -\boldsymbol{F} \cdot d\boldsymbol{l} \,. \tag{15}$$

The choice of $\boldsymbol{a}$ only affects a constant, and the integral must be path-independent. The simplest path turns out to be along $\hat{\boldsymbol{z}}$. Integrating up we find

$$V(x, y, z) = -\sin(z) \ln(x + y^2) \,. \tag{16}$$

It is straightforward to verify that $-\nabla V = \boldsymbol{F}$.

(c) Evaluate the path integral

$$\int_{\mathcal{P}} \boldsymbol{F} \cdot d\boldsymbol{l}\,, \tag{17}$$

along the path $\mathcal{P}$ parameterized by $\boldsymbol{\gamma}(\lambda) = (1+\lambda^2, \sin(\frac{\pi}{2}\lambda), \lambda + \lambda^3)$ from $\lambda = 0$ to $\lambda = 1$. Hint: there is a hard way to evaluate the integral in Eq. (17), and there is an easy way!

Solution: The easy way is to use the divergence theorem, that

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{F} \cdot d\boldsymbol{l} = -V(\boldsymbol{b}) + V(\boldsymbol{a})\,. \tag{18}$$

The points are $\boldsymbol{a} = \boldsymbol{\gamma}(0) = (1,0,0)$ and $\boldsymbol{b} = \boldsymbol{\gamma}(1) = (2,1,2)$. Evaluating V at these two points and taking the difference we get

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{F} \cdot d\boldsymbol{l} = -(-\sin(2)\ln(3)) - 0 = \sin(2)\ln(3)\,. \tag{19}$$

4. For this problem we will work in 3-dimensional cylindrical coordinates $\{s, \phi, z\}$. Consider the vector field

$$\boldsymbol{V} = \frac{s^2}{D}\sin\left(\frac{\pi z}{D}\right)\hat{\boldsymbol{s}} + \frac{s}{\pi}\cos\left(\frac{\pi z}{D}\right)\hat{\boldsymbol{z}}\,, \tag{20}$$

where D is some positive length. Let the volume $\mathcal{V}$ lie in the bounds $0 \le z \le D$, $0 \le \phi \le 2\pi$, and $D \le s \le 2D$.

(a) Evaluate the flux of $\boldsymbol{V}$ going out through the surface $\mathcal{S} = \partial\mathcal{V}$, by performing the relevant surface integrals. Make sure to use the area elements for cylindrical coordinates!

Solution: We will evaluate the integral

$$I = \int_{\partial\mathcal{V}} \boldsymbol{V} \cdot d\boldsymbol{a}\,, \tag{21}$$

where the volume $\partial\mathcal{V}$ is a hollow cylinder of height D, with the inner hollow having radius D and the outer radius being $2D$. There are 4 surfaces: (1) top, (2) bottom, (3) outside, and (4) inside.

i. The top surface, at $z = D$, has directed area element $d\boldsymbol{a} = +\hat{\boldsymbol{z}}\, s\, ds\, d\phi$. This part of the integral is

$$I_1 = \int_D^{2D} ds \int_0^{2\pi} s d\phi \frac{s}{\pi}\cos\left(\frac{\pi z}{D}\right) = -2\pi \frac{s^3}{3\pi}\Bigg|_D^{2D} = -\frac{14D^3}{3}\,. \tag{22}$$

ii. The bottom surface, at $z = 0$. Remember that the directed area element must point outward, so we have $d\boldsymbol{a} = -\hat{\boldsymbol{z}}\, s\, ds\, d\phi$. This part of the integral is

$$I_2 = \int_D^{2D} ds \int_0^{2\pi} s d\phi (-1)\frac{s}{\pi}\cos\left(\frac{\pi z}{D}\right) = -2\pi \frac{s^3}{3\pi}\Bigg|_D^{2D} = -\frac{14D^3}{3}\,. \tag{23}$$

iii. The outside, at $s = 2D$, has directed area element $d\boldsymbol{a} = +\hat{\boldsymbol{s}}\, dz\, s\, d\phi$. This part of the integral is

$$I_3 = \int_0^{D} dz \int_0^{2\pi} s d\phi \frac{s^2}{D}\sin\left(\frac{\pi z}{D}\right) = 2\pi \frac{8D^3}{D}(-\cos\frac{\pi z}{D})\frac{D}{\pi}\Bigg|_0^{D} = 32D^3\,. \tag{24}$$

iv. The inside, at $s = D$. The interior of the volume is between $D < s < 2D$, so to point away from the interior, the directed area element is $d\boldsymbol{a} = -\hat{\boldsymbol{s}}\, dz\, s\, d\phi$. This part of the integral is

$$I_4 = \int_0^{D} dz \int_0^{2\pi} s d\phi (-1)\frac{s^2}{D}\sin\left(\frac{\pi z}{D}\right) = -2\pi \frac{D^3}{D}(-\cos\frac{\pi z}{D})\frac{D}{\pi}\Bigg|_0^{D} = -4D^3\,. \tag{25}$$

Putting these all together we find

$$I = I_1 + I_2 + I_3 + I_4 = \frac{56D^3}{3}\,. \tag{26}$$

(b) Use the divergence theorem to perform the volume integral that should give the same result. Remember to use the cylindrical coordinates form of the divergence, and the correct volume element in the integral!

Solution: The cylindrical coordinates form of the divergence is

$$\nabla \cdot \boldsymbol{V} = \frac{1}{s}\frac{\partial (sV_s)}{\partial s} + \frac{1}{s}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}\,. \tag{27}$$

For our vector field, we have

$$\nabla \cdot \boldsymbol{V} = \frac{2s}{D}\sin\left(\frac{\pi z}{D}\right)\,. \tag{28}$$

Now we need to do the volume integral

$$I = \int (\nabla \cdot \boldsymbol{V}) d^3\mathrm{Vol} = \int_0^D dz \int_D^{2D} ds \int_0^{2\pi} s\, d\phi\, \frac{2s}{D}\sin\left(\frac{\pi z}{D}\right) = \frac{56D^3}{3}\,. \tag{29}$$

5. **Some delta function fun**

(a) Evaluate the integral

$$\int_{-\infty}^{+\infty} \sin(\alpha x)\delta(kx - \omega t)dx\,, \tag{30}$$

where α, k, ω are all real positive constants.

Solution: If we perform a substitution $u = xk - \omega t$, then the argument of the delta function will simply be u and the integral easy to evaluate. For this substitution, $x = (u + \omega t)/k$ and $dx = du/k$; and the integration region is still the whole real line. Thus the original integral is

$$\int_{-\infty}^{+\infty} \sin\left(\frac{\alpha(u+\omega t)}{k}\right)\delta(u)\frac{du}{k}\,. \tag{31}$$

Since there is only a contribution at $u = 0$ we get the value of the integral,

$$\int_{-\infty}^{+\infty} \sin(\alpha x)\delta(kx - \omega t)dx = \sin\left(\frac{\alpha\omega t}{k}\right)\frac{1}{k}\,. \tag{32}$$

(b) Use the fact that $\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$ to evaluate the integral

$$\int_{\text{all space}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot \left(\nabla \frac{\sin(r)}{r}\right) d^3\mathrm{Vol}\,, \tag{33}$$

where $d^3\mathrm{Vol}$ is integrating over $\boldsymbol{r}$, not over $\boldsymbol{r}'$ (I would have put a prime on the volume element if I wanted to denote that).

Solution: We want to move the derivative onto the factor $\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$ in order to get a delta function. To do this we need to integrate by parts, starting with the Leibniz rule

$$\nabla \cdot (f\boldsymbol{V}) = f\nabla \cdot \boldsymbol{V} + \boldsymbol{V} \cdot \nabla f\,. \tag{34}$$

Now integrate over a volume and use the divergence theorem,

$$\int_{\mathcal{V}} \boldsymbol{V} \cdot \nabla f\, d^3\mathrm{Vol} = -\int_{\mathcal{V}} f(\nabla \cdot \boldsymbol{V})\, d^3\mathrm{Vol} + \int_{\partial\mathcal{V}} f\boldsymbol{V} \cdot d\boldsymbol{a}\,. \tag{35}$$

We are going to apply this with $\boldsymbol{V} = \hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$, $f = \sin(r)/r$, and $\mathcal{V} =$ all space (which really means taking the limit as $R \to \infty$ of an ever-growing sphere). Plugging in we find

$$\int_{\text{all space}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot \left(\nabla \frac{\sin(r)}{r}\right) d^3\text{Vol} = -\int_{\text{all space}} \frac{\sin(r)}{r} \nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d^3\text{Vol} + \int_{\partial(\text{all space})} \frac{\sin(r)}{r} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot d\boldsymbol{a}\,. \tag{36}$$

For the first term on the RHS we use the identity $\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$ and can immediately evaluate the integral. We need to check to see if the limit for the second term exists. Since $\sin(r)/r$ decays like $1/r$, and $\hat{\boldsymbol{\mathscr{r}}}/r^2$ decays like $1/r^2$, while the area grows as r^2, the overall integral should be bounded by $1/R$. As $R \to \infty$, the integral converges to 0, so only the first term contributes. Thus we find

$$\int_{\text{all space}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot \left(\nabla \frac{\sin(r)}{r}\right) d^3\text{Vol} = -4\pi \frac{\sin(r')}{r'}\,. \tag{37}$$