UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism I (Phys. 401) — Prof. Leo C. Stein — Fall 2019

Problem Set 3

Due: Friday, Sept. 20, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. **Some "integration by parts" formulas**

   (a) The following is an identity for differentiable functions f and differentiable vector fields $\boldsymbol{V}$:

   $$\int_{\mathcal{S}} f(\nabla \times \boldsymbol{V}) \cdot d\boldsymbol{a} = \int_{\mathcal{S}} (\boldsymbol{V} \times \nabla f) \cdot d\boldsymbol{a} + \oint_{\partial\mathcal{S}} f\boldsymbol{V} \cdot d\boldsymbol{l}\,. \tag{1}$$

   Which "Leibniz rule" (product rule) leads to this integral identity? Starting with that rule, and one of the fundamental theorems, prove the above identity.

   (b) The following is an identity for two differentiable vector fields $\boldsymbol{V}, \boldsymbol{W}$:

   $$\int_{\mathcal{V}} \boldsymbol{V} \cdot (\nabla \times \boldsymbol{W})\, d^3\mathrm{Vol} = \int_{\mathcal{V}} \boldsymbol{W} \cdot (\nabla \times \boldsymbol{V})\, d^3\mathrm{Vol} + \oint_{\partial\mathcal{V}} (\boldsymbol{W} \times \boldsymbol{V}) \cdot d\boldsymbol{a}\,. \tag{2}$$

   Which "Leibniz rule" (product rule) leads to this integral identity? Starting with that rule, and one of the fundamental theorems, prove the above identity.

2. (a) Recall that $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$, where $\boldsymbol{r}'$ is some other fixed point. Show the identity

   $$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,, \tag{3}$$

   where the gradient ∇ just takes derivatives with respect to $\boldsymbol{r} = (x, y, z)$.

   (b) Find the next derivative – now we have to use index notation. We want the whole tensor of second derivatives,

   $$\nabla_i \nabla_j \left(\frac{1}{\mathscr{r}}\right)\,, \tag{4}$$

   which you can think of as a 3x3 matrix. But, you can find a much more compact way to write it in terms of $\boldsymbol{\mathscr{r}}_k$.

3. **When a vector field is conservative...** Consider the vector field

   $$\boldsymbol{F} = \frac{\sin z}{x + y^2}\hat{\boldsymbol{x}} + \frac{2y \sin z}{x + y^2}\hat{\boldsymbol{y}} + \cos(z)\ln(x + y^2)\hat{\boldsymbol{z}}\,. \tag{5}$$

   (a) Show that this vector field is conservative.

   (b) Find a scalar potential (up to an additive constant) such that $\boldsymbol{F} = -\nabla V$.

   (c) Evaluate the path integral

   $$\int_{\mathcal{P}} \boldsymbol{F} \cdot d\boldsymbol{l}\,, \tag{6}$$

   along the path $\mathcal{P}$ parameterized by $\boldsymbol{\gamma}(\lambda) = (1 + \lambda^2, \sin(\frac{\pi}{2}\lambda), \lambda + \lambda^3)$ from $\lambda = 0$ to $\lambda = 1$. Hint: there is a hard way to evaluate the integral in Eq. (6), and there is an easy way!

4. For this problem we will work in 3-dimensional cylindrical coordinates $\{s, \phi, z\}$. Consider the vector field

$$\boldsymbol{V} = \frac{s^2}{D} \sin\left(\frac{\pi z}{D}\right) \hat{\boldsymbol{s}} + \frac{s}{\pi} \cos\left(\frac{\pi z}{D}\right) \hat{\boldsymbol{z}}\,, \tag{7}$$

where D is some positive length. Let the volume $\mathcal{V}$ lie in the bounds $0 \leq z \leq D$, $0 \leq \phi \leq 2\pi$, and $D \leq s \leq 2D$.

   (a) Evaluate the flux of $\boldsymbol{V}$ going out through the surface $\mathcal{S} = \partial\mathcal{V}$, by performing the relevant surface integrals. Make sure to use the area elements for cylindrical coordinates!

   (b) Use the divergence theorem to perform the volume integral that should give the same result. Remember to use the cylindrical coordinates form of the divergence, and the correct volume element in the integral!

5. **Some delta function fun**

   (a) Evaluate the integral

$$\int_{-\infty}^{+\infty} \sin(\alpha x)\delta(kx - \omega t)dx\,, \tag{8}$$

   where α, k, ω are all real positive constants.

   (b) Use the fact that $\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$ to evaluate the integral

$$\int_{\text{all space}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot \left(\nabla \frac{\sin(r)}{r}\right) d^3\text{Vol}\,, \tag{9}$$

   where $d^3\text{Vol}$ is integrating over $\boldsymbol{r}$, not over $\boldsymbol{r}'$ (I would have put a prime on the volume element if I wanted to denote that).