UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism I (Phys. 401) — Prof. Leo C. Stein — Fall 2019

Problem Set 4 — SOLUTIONS

Due: Friday, Sept. 27, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. Suppose we create the following charge distribution,

$$\rho(r,\theta,\phi) = \begin{cases} \dfrac{\kappa}{r^2} & R_1 < r < R_2 \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where κ is some constant. Use Gauss's law to find the $\boldsymbol{E}$ field everywhere in space.

Solution: Because of spherical symmetry, the only nonvanishing component of the electric field is E_r, so $\boldsymbol{E} = E_r\hat{\boldsymbol{r}}$. To find the dependence $E_r(r)$, we will use the integral form of Gauss's law on a sphere of radius r centered at the origin. Let's call this surface S_r^2. So, the value of E_r is found from

$$\int_{S_r^2} \boldsymbol{E}\cdot d\boldsymbol{a} = \frac{Q_{\text{enc}}(r)}{\epsilon_0}\,, \tag{2}$$

$$4\pi r^2 E_r = \frac{Q_{\text{enc}}(r)}{\epsilon_0}\,, \tag{3}$$

$$E_r = \frac{Q_{\text{enc}}(r)}{4\pi r^2\epsilon_0}\,. \tag{4}$$

Now we need to determine how much charge there is interior on the interior of this surface, that is inside the ball of radius r, named B_r^3. The charge enclosed is

$$Q_{\text{enc}}(r) = \int_{B_r^3} \rho(\boldsymbol{r}')\, d^3\text{Vol}'\,. \tag{5}$$

We have three cases for this integral: (i) when $r < R_1$, (ii) when $R_1 \le r \le R_2$, and (iii) when $R_2 < 3$. For case (i), the charge enclosed is 0. For case (ii),

$$Q_{\text{enc}}(r) = \int_0^r \rho(\boldsymbol{r}')(r')^2 dr d\cos\theta d\phi \tag{6}$$

$$= 4\pi\int_0^r \rho(r) r^2 dr = 4\pi\int_{R_1}^r \kappa\, dr \tag{7}$$

$$Q_{\text{enc}}(r) = 4\pi\kappa(r - R_1) \qquad \text{(when } R_1 \le r \le R_2\text{).} \tag{8}$$

Finally, when $R_2 < r$, the only charge that is contributing is that from R_1 to R_2, so the charge enclosed stops growing,

$$Q_{\text{enc}}(r) = 4\pi\kappa(R_2 - R_1) \qquad \text{(when } R_2 \le r\text{).} \tag{9}$$

Now putting this together we get the total electric field, $\boldsymbol{E} = E_r\hat{\boldsymbol{r}}$, where

$$E_r = \begin{cases} 0 & r < R_1 \\ \dfrac{\kappa(r-R_1)}{r^2\epsilon_0} & R_1 \le r \le R_2 \\ \dfrac{\kappa(R_2-R_1)}{r^2\epsilon_0} & R_2 < r\,. \end{cases} \tag{10}$$

2. What is the $\boldsymbol{E}$ field produced by an infinite slab that stretches in the x, y directions and is restricted to $-h \leq z \leq +h$ for some positive h, with uniform charge density ρ_0? Find a potential V that corresponds to this $\boldsymbol{E}$.

   Solution: Because of the translation and rotation symmetry in the $x-y$ plane, the $\boldsymbol{E}$ field can only have a z component, $\boldsymbol{E} = E_z \hat{\boldsymbol{z}}$, and the component can only depend on z itself, $E_z = E_z(z)$.

   Furthermore because of the reflection symmetry across the $x-y$ plane, we should have $E_z(-z) = -E_z(z)$. In words, the z component of the $\boldsymbol{E}$ field at a height z above the plane and below the plane have the same magnitude, but opposite signs.

   Therefore we will choose as an integration region of height $2z$, extending from $-z$ to $+z$, and the shape in the $x-y$ directions does not matter (as long as it's the same shape at every value of z). It is simplest to choose a square of area A in the $x-y$ plane. The flux of $\boldsymbol{E}$ through this rectangular prism $\mathcal{V}$ will only be through the top and bottom faces, of area A.

   Now there are two cases for the integration: (i) either $2z \leq 2h$, or (ii) $2h < 2z$. In the first case, the integral form of Gauss's law tells us

$$\int_{\mathcal{V}} \boldsymbol{E} \cdot d\boldsymbol{a} = 2AE_z(z) = \frac{Q_{\text{enc}}}{\epsilon_0} \tag{11}$$

$$2AE_z(z) = A \times 2d \times \frac{\rho_0}{\epsilon_0} \tag{12}$$

$$E_z(z) = \frac{d\rho_0}{\epsilon_0} \qquad \text{(when } 0 \leq z \leq h\text{)}. \tag{13}$$

   In the second case, the total amount of charge enclosed comes from the full thickness of the sheet,

$$E_z(z) = \frac{h\rho_0}{\epsilon_0} \qquad \text{(when } h \leq z\text{)}. \tag{14}$$

   From the antisymmetry, $E_z = -h\rho_0/\epsilon_0$ for $z < -h$.

   Now we look for a potential whose negative gradient gives this field, $\boldsymbol{E} = -\nabla V$. It's enough for this potential to only depend on z. The simplest potential will have $V(z=0) = 0$. If we integrate the above electric field along a straight path from $z = 0$ to some $z = z$ we can find the desired potential,

$$V(z) = \begin{cases} \dfrac{(2z+h)h\rho_0}{2\epsilon_0} & z < -h \\ -\dfrac{z^2\rho_0}{2\epsilon_0} & -h \leq z \leq +h \\ \dfrac{(h-2z)h\rho_0}{2\epsilon_0} & +h < z\,. \end{cases} \tag{15}$$

3. Now we distribute charge along the line segment $y = 0, z = 0$, between $-L \leq x \leq +L$ for some positive L. But we do not distribute the charge uniformly; instead we apply the charge density per unit length $\lambda(x') = \lambda_0 x'/L$. Find the potential $V(x, y, z)$ created by this charge distribution. You may find the following antiderivatives helpful:

$$\int \frac{du}{\sqrt{u^2+b}} = \log(u + \sqrt{u^2+b})\,, \qquad \int \frac{u\,du}{\sqrt{u^2+b}} = \sqrt{u^2+b}\,. \tag{16}$$

   Then find the $\boldsymbol{E}$ field everywhere.

   Solution: Since the charge distribution is azimuthally symmetric about the line (i.e. about the x axis), we know that the potential will be as well. Therefore it is only a function of x and the distance away from the x axis, i.e.

$$V = V(s, x)\,, \tag{17}$$

where $s^2 = y^2 + z^2$. Ideally we would use a cylindrical coordinate system with the symmetry axis along x instead of the usual z. Or, we could rotate the charge configuration so that it lies along z, find the electric potential V and the field $\boldsymbol{E}$, and then rotate back.

In any case, the potential is found from

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r}')}{\mathscr{r}} d^3\mathrm{Vol}' \tag{18}$$

$$V(s,x) = \frac{1}{4\pi\epsilon_0} \int_{-L}^{+L} \frac{\lambda_0 x'}{L\sqrt{(x-x')^2+s^2}} dx' \,. \tag{19}$$

Now using the provided antiderivatives, and a simply substitution,

$$\int \frac{x'\,dx'}{\sqrt{(x-x')^2+s^2}} = \sqrt{(x-x')^2+s^2} + x\log\left[(x'-x) + \sqrt{(x-x')^2+s^2}\right] + C\,. \tag{20}$$

Evaluating at the two endpoints, we find

$$V(s,x) = \frac{\lambda_0}{4\pi L\epsilon_0}\left\{\sqrt{(x-L)^2+s^2} - \sqrt{(x+L)^2+s^2} + x\log\left[\frac{(L-x)+\sqrt{(x-L)^2+s^2}}{(-L-x)+\sqrt{(x+L)^2+s^2}}\right]\right\} \tag{21}$$

Now to find the electric field, we need $\boldsymbol{E} = -\nabla V$. We will make a lot of use of the chain rule. For example,

$$\nabla s = \nabla\sqrt{y^2+z^2} = \frac{2y\hat{\boldsymbol{y}} + 2z\hat{\boldsymbol{z}}}{2\sqrt{y^2+z^2}} \equiv \hat{\boldsymbol{s}}\,. \tag{22}$$

You can check that this is indeed a unit vector, the norm being $(y^2+z^2)/s^2 = 1$. Again applying the chain rule we have

$$\nabla\sqrt{(x-L)^2+s^2} = \frac{(x-L)\hat{\boldsymbol{x}} + s\hat{\boldsymbol{s}}}{\sqrt{(x-L)^2+s^2}} \tag{23}$$

and similarly for

$$\nabla\sqrt{(x+L)^2+s^2} = \frac{(x+L)\hat{\boldsymbol{x}} + s\hat{\boldsymbol{s}}}{\sqrt{(x+L)^2+s^2}}\,. \tag{24}$$

Continuing on we have

$$\begin{aligned}\nabla\left\{x\log\left[(L-x)+\sqrt{(x-L)^2+s^2}\right]\right\} &= \hat{\boldsymbol{x}}\log\left[(L-x)+\sqrt{(x-L)^2+s^2}\right] \qquad (25)\\ &\quad + \frac{x}{(L-x)+\sqrt{(x-L)^2+s^2}}\left[-\hat{\boldsymbol{x}} + \frac{(x-L)\hat{\boldsymbol{x}}+s\hat{\boldsymbol{s}}}{\sqrt{(x-L)^2+s^2}}\right]\end{aligned}$$

where we get to reuse one of the earlier results; and similarly for

$$\begin{aligned}\nabla\left\{x\log\left[(-L-x)+\sqrt{(x+L)^2+s^2}\right]\right\} &= \hat{\boldsymbol{x}}\log\left[(-L-x)+\sqrt{(x+L)^2+s^2}\right] \qquad (26)\\ &\quad + \frac{x}{(-L-x)+\sqrt{(x+L)^2+s^2}}\left[-\hat{\boldsymbol{x}} + \frac{(x+L)\hat{\boldsymbol{x}}+s\hat{\boldsymbol{s}}}{\sqrt{(x+L)^2+s^2}}\right]\end{aligned}$$

The final result is the sum of these four pieces times a constant,

$$\begin{aligned}\boldsymbol{E} = -\nabla V = \frac{-\lambda_0}{4\pi L\epsilon_0}\Bigg[&\nabla\sqrt{(x-L)^2+s^2} - \nabla\sqrt{(x+L)^2+s^2} \qquad (27)\\ &+ \nabla\left\{x\log\left[(L-x)+\sqrt{(x-L)^2+s^2}\right]\right\} \qquad (28)\\ &- \nabla\left\{x\log\left[(-L-x)+\sqrt{(x+L)^2+s^2}\right]\right\}\Bigg]\,, \qquad (29)\end{aligned}$$

using the above results.

4. Let's distribute a total charge q throughout the half-sphere that lies in the region $x^2 + y^2 + z^2 \leq R$ and $z \leq 0$. Find the potential $V(0, 0, z)$ with $z > 0$ along the positive z axis. Now you might find these additional antiderivatives helpful:

$$\int \frac{\sin\theta \, d\theta}{\sqrt{b + c\cos\theta}} = -\frac{2}{c}\sqrt{b + c\cos\theta}\,, \tag{30}$$

$$\int \sqrt{u^2 + b}\, du = \frac{u}{2}\sqrt{u^2 + b} + \frac{b}{2}\log(u + \sqrt{u^2 + b})\,, \tag{31}$$

$$\int u\sqrt{u^2 + b}\, du = \frac{1}{3}(u^2 + b)^{3/2}\,. \tag{32}$$

Solution: We want to use the integral solution for the potential,

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}')}{\mathscr{r}} d^3\text{Vol}'\,, \tag{33}$$

but we only need to evaluate this integral at $\boldsymbol{r} = (0, 0, z)$ where $z > 0$. The source region is easily expressed in spherical polar coordinates as $0 \leq r \leq R$, $\pi/2 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. Thus our integral is

$$V(0, 0, z) = \frac{\rho}{4\pi\epsilon_0}\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^R \frac{1}{\sqrt{(z - r'\cos\theta')^2 + (r'\sin\theta')^2}}(r')^2 dr' \sin\theta' d\theta' d\phi'\,, \tag{34}$$

where $\rho = 3q/(2\pi R^3)$ is the uniform charge density. Notice that nothing depends on ϕ' (because we only evaluate on the axis), so we can immediately do the ϕ' integral. What remains to evaluate is

$$V(0, 0, z) = \frac{\rho}{2\epsilon_0}\int_{\pi/2}^{\pi}\int_0^R \frac{1}{\sqrt{z^2 - 2zr'\cos\theta' + (r')^2}}(r')^2 dr' \sin\theta' d\theta'\,, \tag{35}$$

First doing the θ' integral, with $b = z^2 + (r')^2$ and $c = -2zr'$, we get

$$V(0, 0, z) = \frac{\rho}{2\epsilon_0}\int_0^R \left[(z + r') - \sqrt{z^2 + (r')^2}\right]\frac{r'}{z} dr'\,. \tag{36}$$

Now to do the r' integral we have one term that's a polynomial in r', and one term of the form $\int r'\sqrt{(r')^2 + b} dr'$, for which we have an antiderivative above. Integrating between the two endpoints we finally find

$$V(0, 0, z) = \frac{\rho}{2\epsilon_0}\frac{1}{z}\left[\frac{1}{3}R^3 + \frac{1}{2}R^2 z + \frac{1}{3}z^3 - \frac{1}{3}(R^2 + z^2)^{3/2}\right]\,. \tag{37}$$