UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism I (Phys. 401) — Prof. Leo C. Stein — Fall 2019

Problem Set 5 — SOLUTIONS

Due: Monday, Oct. 7, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. **Surface charge on a non-uniform disk**. We've arranged charge to lie on a disk of radius R in the $z = 0$ plane. The surface charge density for a radius s away from the origin, where $0 \le s \le R$, is

$$\sigma(s) = \sigma_0 + \frac{s}{R}(\sigma_R - \sigma_0)\,, \tag{1}$$

where σ_0 is the surface charge density at the center, and σ_R is the surface charge density at the edge. Find the potential $V(0, 0, z)$ along the z axis at some height z above the disk. (If you look up an integral in a table such as Gradshteyn and Ryzhik, include the reference to the identity you used; Mathematica or another computer algebra system will not count.)

Solution: We use the integral form of the solution for the potential, and only need to evaluate at $\boldsymbol{r} = (0, 0, z)$,

$$V(0,0,z) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}')}{\mathscr{r}} d^3\mathrm{Vol}'\,. \tag{2}$$

The charge is restricted to the plane $z' = 0$ and for $0 \le s' \le R$. Since we are on the axis, the integrand is azimuthally symmetric, so we can do the ϕ' integral right away. So, the integral is

$$V(0,0,z) = \frac{1}{2\epsilon_0}\int_0^R \frac{\sigma_0 + (\sigma_R - \sigma_0)s'/R}{\sqrt{z^2 + (s')^2}} s'\,ds'\,. \tag{3}$$

We need three integrals, two of which were given in last's week problem set,

$$\int \frac{du}{\sqrt{u^2+b}} = \ln(u + \sqrt{u^2+b})\,, \qquad \int \frac{u\,du}{\sqrt{u^2+b}} = \sqrt{u^2+b}\,. \tag{4}$$

The third one can be found in an integral table, e.g. in Gradshteyn and Rhyzik 7th edition, formula 2.272.3,

$$\int \frac{x^2}{u}dx = \frac{1}{2}\frac{xu}{c} - \frac{1}{2}\frac{a}{c}\frac{1}{\sqrt{c}}\ln(x\sqrt{c} + u)\,, \tag{5}$$

where $u = \sqrt{a + cx^2}$.

Using these identities we get

$$V(0,0,z) = \frac{1}{2\epsilon_0}\left[-z\sigma_0 + \frac{1}{2}(\sigma_0 + \sigma_R)\sqrt{z^2+R^2} + \frac{(\sigma_0 - \sigma_R)z^2}{2R}\ln\frac{R + \sqrt{z^2+R^2}}{z}\right] \tag{6}$$

2. **Cone of charge** (Griffiths 2.26 in 3rd ed.). We place a uniform surface charge density σ on the surface of a cone (like a hollow ice cream cone, with no top). The cone has a height h and the radius at the top is also h. Find the potential difference between point $\boldsymbol{a}$ (the vertex) and point $\boldsymbol{b}$ (the center of the top).

Solution: We will evaluate $V(\boldsymbol{a})$ and $V(\boldsymbol{b})$ directly from the integral

$$V(\boldsymbol{r}) = \frac{1}{4p\epsilon_0} \int \frac{\rho(\boldsymbol{r'})}{\mathscr{r}} d^3\text{Vol}\,. \tag{7}$$

This is made a bit simpler because both $\boldsymbol{a}$ and $\boldsymbol{b}$ are on the symmetry axis, so the two-dimensional surface integral can be reduced to a one-dimensional integral.

Let's start with $\boldsymbol{a}$. Suppose we put the vertex of the cone at the origin, and use cylindrical coordinates. Let's use z as the parameter to integrate. At each z, there is an infinitesimal area of $2\pi z\sqrt{2}dz$, carrying surface charge density σ. The $\sqrt{2}$ comes from the angle of the cone. These points are at distance $\mathscr{r} = \sqrt{2}z$ away. The potential is thus

$$V(\boldsymbol{a}) = \frac{1}{4\pi\epsilon_0} \int_0^h \frac{\sigma 2\pi z\sqrt{2}dz}{\sqrt{2}z} = \frac{h\sigma}{2\epsilon_0}\,. \tag{8}$$

Now on to $\boldsymbol{b}$. Again using z as the parameter, we have the same infinitesimal area element and charge. But now the points at coordinate z are a distance $\mathscr{r} = \sqrt{z^2 + (h-z)^2}$ away. Thus we need to integrate

$$V(\boldsymbol{b}) = \frac{1}{4\pi\epsilon_0} \int_0^h \frac{\sigma 2\pi z\sqrt{2}dz}{\sqrt{z^2 + (h-z)^2}} = \frac{\sigma}{\epsilon_0\sqrt{2}} \int_0^h \frac{zdz}{\sqrt{z^2 + (h-z)^2}}\,. \tag{9}$$

To do the integral, substitute $u = z - h/2$, so we get

$$V(\boldsymbol{b}) = \frac{\sigma}{2\epsilon_0} \int_{-h/2}^{+h/2} \frac{u + \frac{h}{2}}{\sqrt{u^2 + h^2/4}} du\,. \tag{10}$$

This integral can be done with two of the identities above. The result is

$$V(\boldsymbol{b}) = \frac{\sigma}{2\epsilon_0} \left\{ \frac{1}{2}\left[\sqrt{h^2 + 4u^2} + h\ln\left(2u + \sqrt{h^2 + 4u^2}\right)\right]\right\}_{-h/2}^{+h/2}\,. \tag{11}$$

After evaluating at the two endpoints we have

$$V(\boldsymbol{b}) = \frac{h\sigma}{4\epsilon_0}\ln(3 + 2\sqrt{2}) = \frac{h\sigma}{2\epsilon_0}\ln(1 + \sqrt{2})\,. \tag{12}$$

Finally taking the difference we get

$$V(\boldsymbol{a}) - V(\boldsymbol{b}) = \frac{h\sigma}{2\epsilon_0}\left(1 - \ln(1 + \sqrt{2})\right)\,. \tag{13}$$

3. **Energy of a non-uniform spherical charge distribution**. Distribute charge onto a ball of radius R so that the charge density is

$$\rho(r) = \begin{cases} \rho_R \frac{r^2}{R^2} & 0 \le r \le R \\ 0 & \text{otherwise}\,, \end{cases} \tag{14}$$

where ρ_R is the volume charge density at the edge.

(a) Compute the potential V and electric field $\boldsymbol{E}$ everywhere.

Solution: Because of the spherical symmetry, the electric field is purely radial, $\boldsymbol{E} = E_r\hat{\boldsymbol{r}}$. To find E_r we can use Gauss's law on spherical surfaces of radii r. When $r \le R$, we have

$$\int \boldsymbol{E} \cdot d\boldsymbol{a} = 4\pi r^2 E_r(r) = \frac{Q_{\text{enc}}(r)}{\epsilon_0}\,. \tag{15}$$

The enclosed charge is

$$Q_{\rm enc}(r) = \int \rho(\boldsymbol{r}') d^3{\rm Vol}' = \int_0^r \rho_R \frac{(r')^2}{R^2} 4\pi (r')^2 dr' \tag{16}$$

$$Q_{\rm enc}(r) = \frac{4\pi\rho_R}{R^2} \int_0^r (r')^4 dr' = \frac{4\pi\rho_R}{R^2}\frac{r^5}{5}\,. \tag{17}$$

As a quick check, the units of $Q_{\rm enc}$ are indeed those of charge. If we evaluate this at $r = R$ then we find the total charge is $Q_{\rm tot} = 4\pi R^3 \rho_R/5$.
So, for $r \leq R$, we have the electric field

$$E_r(r \leq R) = \frac{Q_{\rm enc}(r)}{4\pi\epsilon_0 r^2} = \frac{\rho_R r^3}{5\epsilon_0 R^2}\,. \tag{18}$$

Exterior, when $R < r$, we instead find

$$E_r(R < r) = \frac{Q_{\rm tot}}{4\pi\epsilon_0 r^2} = \frac{\rho_R R^3}{5\epsilon_0 r^2}\,. \tag{19}$$

Now let's integrate the electric field to get the potential. Since the electric field is radial and spherically symmetric, the potential will be a function of only r. In the exterior, we have

$$V(R < r) = \frac{Q_{\rm tot}}{4\pi\epsilon_0 r} = \frac{\rho_R R^3}{5\epsilon_0 r}\,. \tag{20}$$

And, in the interior, we get

$$V(r \leq R) = -\frac{\rho_R r^4}{20\epsilon_0 R^2} + \frac{\rho_R R^2}{20\epsilon_0} + \frac{\rho_R R^2}{5\epsilon_0} = -\frac{\rho_R r^4}{20\epsilon_0 R^2} + \frac{\rho_R R^2}{4\epsilon_0}\,. \tag{21}$$

(b) Compute the energy of this charge distribution, using V and ρ.
Solution: We need to integrate

$$W = \frac{1}{2}\int \rho V d^3{\rm Vol}\,, \tag{22}$$

where ρ is only non-vanishing for $r < R$. Since everything is spherically symmetric we can do the angular integral right away. Now we have to evaluate

$$W = \frac{1}{2}\int_0^R \rho_R \frac{r^2}{R^2}\left[-\frac{\rho_R r^4}{20\epsilon_0 R^2} + \frac{\rho_R R^2}{4\epsilon_0}\right] 4\pi r^2\, dr\,. \tag{23}$$

This is all polynomial in r. We find

$$W = \frac{4\pi\rho_R^2 R^5}{45\epsilon_0}\,. \tag{24}$$

(c) Compute the energy of this charge distribution, using $\boldsymbol{E}$.
Solution: Now we have to integrate

$$W = \frac{\epsilon_0}{2}\int E^2\, d^3{\rm Vol}\,, \tag{25}$$

and the integral will extend over all space because the $\boldsymbol{E}$ field is nowhere vanishing. The norm squared of the $\boldsymbol{E}$ field is $E^2 = E_r(r)^2$ and is only a function of r, so we can do the angular integral. We need to compute

$$W = \frac{\epsilon_0}{2}\int_0^\infty E_r^2\, 4\pi r^2\, dr\,. \tag{26}$$

We split the integration region into the interior and exterior parts,

$$W = \frac{4\pi\epsilon_0}{2}\left\{\int_0^R \left(\frac{\rho_R r^3}{5\epsilon_0 R^2}\right)^2 r^2\,dr + \int_R^\infty \left(\frac{\rho_R R^3}{5\epsilon_0 r^2}\right)^2 r^2\,dr\right\} \tag{27}$$

These are only power laws in r so also easy to integrate. We again find

$$W = \frac{4\pi\rho_R^2 R^5}{45\epsilon_0}\,, \tag{28}$$

which agrees with the other approach to calculating energy.

4. **Two conductors**. We take one solid conducting ball of radius r_1, and place it inside a hollow conducting shell of radius r_2 (with negligible thickness). Now we place a charge $+q$ on the inner ball, and charge $-q$ on the outer shell, and let the charges come to static equilibrium.

   (a) Find the potential V and electric field $\boldsymbol{E}$ everywhere.

   Solution: Since the ball is a conductor, all the charge will move to its surface. Because of spherical symmetry, the charge will distribute itself uniformly, and the electric field will only be radial, $\boldsymbol{E} = E_r\hat{\boldsymbol{r}}$.

   Using Gauss's law, we can find that the electric field vanishes interior to r_1 and exterior to r_2. In between, the electric field will be $E_r = q/(4\pi\epsilon_0 r^2)$.

   We can find the potential $\int \boldsymbol{E}\cdot d\boldsymbol{l}$ along a radial curve that runs from r to ∞, effectively setting the potential to zero at infinity, $V(\infty) = 0$. Therefore we get

$$V(r) = \begin{cases} \dfrac{q}{4\pi\epsilon_0}\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right) & r < r_1 \\ \dfrac{q}{4\pi\epsilon_0}\left(\dfrac{1}{r} - \dfrac{1}{r_2}\right) & r_1 \le r \le r_2 \\ 0 & r_2 < r\,. \end{cases} \tag{29}$$

   (b) Compute the energy required to assemble this charge distribution, by using V and ρ

   Solution: The energy is computed from V and ρ from

$$W = \frac{1}{2}\int \rho V\, d^3\mathrm{Vol}\,. \tag{30}$$

   So, we need to find $\rho(\boldsymbol{r})$ for this configuration. We have a charge $+q$ uniformly distributed on the sphere $r = r_1$, and charge $-q$ uniformly distributed on the sphere $r = r_2$. This corresponds to

$$\rho(\boldsymbol{r}) = \frac{q}{4\pi r_1^2}\delta(r - r_1) - \frac{q}{4\pi r_2^2}\delta(r - r_2)\,. \tag{31}$$

   This the energy integral reduces to integrals over these two spherical surfaces, both of which are spherically symmetric, so we get

$$W = \frac{1}{2}\left[qV(r_1) - qV(r_2)\right]. \tag{32}$$

   Notice that $V(r_2) = 0$. Plugging in the potential from before we find

$$W = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\,. \tag{33}$$

(c) Compute the energy required to assemble this charge distribution, using $\boldsymbol{E}$.

Solution: The energy is computed from the electric field via

$$W = \frac{\epsilon_0}{2} \int E^2 \, d^3\text{Vol} \,. \tag{34}$$

Because of spherical symmetry we can do the angular integrals, and the $\boldsymbol{E}$ field is only non-vanishing between r_1 and r_2, so we have

$$W = \frac{\epsilon_0}{2} 4\pi \int_{r_1}^{r_2} r^2 E_r^2 \, dr = \frac{\epsilon_0}{2} 4\pi \int_{r_1}^{r_2} r^2 \frac{q^2}{(4\pi\epsilon_0)^2 r^4} \, dr \tag{35}$$

$$W = \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) . \tag{36}$$

This agrees with the other method of computing the energy.

(d) Compute the capacitance of this geometry.

Solution: The capacitance is the ratio of stored charge to the potential difference between the two conductors,

$$C = \frac{q}{\Delta V} = 4\pi\epsilon_0 \left(\frac{1}{r_1} - \frac{1}{r_2} \right)^{-1} . \tag{37}$$