UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism I (Phys. 401) — Prof. Leo C. Stein — Fall 2019

Problem Set 6 — SOLUTIONS

Due: Wednesday, Oct. 16, 2019, by 5PM

Material: The midterm covers the material so far (up through and including Griffiths' chapter 3.2).

Due date: Wednesday, Oct. 16, 2019 by 5PM to 205 Lewis Hall. If my door is closed, please slide the exam under my door. Late exams will require extenuating circumstances.

Logistics: The exam consists of this page plus one page of questions. Do not look at the problems until you are ready to start it.

Time: The work might expand to eat up as much time as you allot – therefore I highly recommend you restrict yourself to no more than 5 hours cumulative time spent on these problems. You may take as many breaks as you like, not counted against the 5 hours. **You should not be consulting references, working on the problems, or discussing with others during the breaks**.

Resources: The midterm and final are **not collaborative**. All questions must be done on your own, without consulting anyone else. You may consult your own notes (both in-class and notes on this class you or a colleague in the class have made), the textbook by Griffiths, and solution sets on the course website. **You may not consult any other material**, including other textbooks, the web (except for the current Phys. 401 website), material from previous years' Phys. 401 or any other classes, or copies you have made of such material, or any other resources. Calculators and symbolic manipulation programs are not allowed.

1. **Math warm-up**.

(a) Consider the function f given in cylindrical coordinates,

$$f(s,\phi,z) \equiv z^2 + s^2\left(\frac{1}{9}\cos^2\phi + \frac{1}{4}\sin^2\phi\right) . \tag{1}$$

Describe the surface defined by the level set $f(s,\phi,z) = 1$. Find the unit vector normal to this surface at some arbitrary point.

Solution: Using $s\cos\phi = x$, $s\sin\phi = y$, the function f is

$$f = \frac{x^2}{3^2} + \frac{y^2}{2^2} + \frac{z^2}{1^2} . \tag{2}$$

The equation $f = 1$ then is the standard way of describing an ellipsoid with principal axes of length $(3,2,1)$ in the (x,y,z) directions.

We can get the unit normal by normalizing any vector that's orthogonal to the surface, e.g. the gradient of f. This gradient is

$$\nabla f = 2z\hat{\boldsymbol{z}} + 2s\hat{\boldsymbol{s}}\left(\frac{1}{9}\cos^2\phi + \frac{1}{4}\sin^2\phi\right) + s^2\frac{5}{18}\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} \tag{3}$$

$$= \frac{2x\hat{\boldsymbol{x}}}{3^2} + \frac{2y\hat{\boldsymbol{y}}}{2^2} + \frac{2z\hat{\boldsymbol{z}}}{1^2} . \tag{4}$$

The norm is $|\nabla f|^2 = 4x^2/81 + y^2/4 + 4z^2$. Normalizing,

$$\hat{\boldsymbol{n}}(x,y,z) = \frac{1}{\sqrt{4x^2/81 + y^2/4 + 4z^2}}\left(\frac{2x\hat{\boldsymbol{x}}}{3^2} + \frac{2y\hat{\boldsymbol{y}}}{2^2} + \frac{2z\hat{\boldsymbol{z}}}{1^2}\right) . \tag{5}$$

(b) Take the vector field $\boldsymbol{v}$ to be given in spherical coordinates as

$$\boldsymbol{v} = r\sin\phi\,\hat{\boldsymbol{r}} + 2r\sin\theta\cos\phi\,\hat{\boldsymbol{\theta}} + 3r^2\sin\theta\sin\phi\,\hat{\boldsymbol{\phi}} . \tag{6}$$

Evaluate the flux of $\boldsymbol{v}$ through the following surface $\mathcal{S}$, which is the boundary of the volume where $0 \le r \le 1$, $0 \le \theta \le \pi/3$, and $0 \le \phi \le \pi$.

Solution: In this case, the easiest way to evaluate this flux is to use the divergence theorem to convert it to a volume integral. Thus we need the divergence,

$$\nabla\cdot\boldsymbol{v} = 3r\cos\phi + 4\cos\theta\cos\phi + 3\sin\phi . \tag{7}$$

Now we need to do the volume integral,

$$I = \int_0^1\int_0^{\pi/3}\int_0^{\pi}(\nabla\cdot\boldsymbol{v})\,r^2\sin\theta\,d\phi d\theta dr \tag{8}$$

$$I = 1 . \tag{9}$$

(c) Show that the following vector field is conservative:

$$\boldsymbol{v} = \ln(x^2y + z)\left(2xy\,\hat{\boldsymbol{x}} + x^2\,\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}\right) . \tag{10}$$

Solution: All one needs to do is take the curl, which vanishes. If you want, you can find a potential U where $\boldsymbol{v} = \nabla U$. Up to an additive constant, $U = (x^2y + z)\left(\ln(x^2y + z) - 1\right)$.

2. **Infinite charged cylinder**. Suppose we are so infinitely powerful that we can construct an infinite cylinder centered on the $\hat{z}$ axis, of diameter D. We distribute charge so that the space density is

$$\rho(s,\phi,z) = \begin{cases} \alpha s & \text{inside,} \\ 0 & \text{outside,} \end{cases} \tag{11}$$

where α is some positive constant.

(a) Find the electric field $\boldsymbol{E}$ everywhere (both inside and outside the cylinder).

Solution: Because of the symmetry of the problem, we can use Gauss's law to find the electric field. The electric field is purely in the radial $\hat{\boldsymbol{s}}$ direction. Take a cylinder $\mathcal{V}$ of height h and radius s. When $s < D/2$, the charge enclosed is

$$Q_{\text{enc}}(s < D/2) = \int_0^s (\alpha s')\, 2\pi h s'\, ds' = 2\pi\alpha h \frac{s^3}{3}\,. \tag{12}$$

When $s \geq D/2$, there is no additional enclosed charge, so $Q_{\text{enc}}(s \geq D/2) = \pi\alpha h D^3/12$. In Gauss's law,

$$\oint_{\partial\mathcal{V}} \boldsymbol{E}\cdot d^2\boldsymbol{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \tag{13}$$

$$2\pi s h E_s(s) = \frac{Q_{\text{enc}}}{\epsilon_0}\,, \tag{14}$$

because the only flux is through the sides of the cylinder, nothing from the top/bottom. Now plugging in,

$$\boldsymbol{E} = E_s\hat{\boldsymbol{s}} = \begin{cases} \dfrac{\alpha s^2}{3\epsilon_0}\hat{\boldsymbol{s}}\,, & s < D/2 \\ \dfrac{\alpha D^3}{24\epsilon_0 s}\hat{\boldsymbol{s}}\,, & s \geq D/2\,. \end{cases} \tag{15}$$

(b) Find the electric potential $\boldsymbol{V}$ everywhere (inside and outside).

Solution: Since the electric field only depends on s and only points in the $\hat{\boldsymbol{s}}$ direction, we can get a potential $V(s)$ that only depends on s. We can find it by integrating from some base point $\mathcal{O}$

$$V(\boldsymbol{r}) = -\int_{\mathcal{O}}^{\boldsymbol{r}} \boldsymbol{E}\cdot d\boldsymbol{l}\,. \tag{16}$$

Again since there is only s dependence, perform a radial integral, $d\boldsymbol{l} = \hat{s}\,ds$. Let's choose the base point to be $s = 0$. Performing the integral for $s < D/2$,

$$V(s < D/2) = -\int_0^s \frac{\alpha (s')^2}{3\epsilon_0} ds' = -\frac{\alpha s^3}{9\epsilon_0}\,. \tag{17}$$

Then continuing from the surface onward out,

$$V(s \geq D/2) = -\frac{\alpha D^3}{8\times 9\epsilon_0} - \int_{D/2}^{s} \frac{\alpha D^3}{24\epsilon_0 s} ds' = -\frac{\alpha D^3}{8\times 9\epsilon_0} - \frac{\alpha D^3}{24\epsilon_0}\ln\frac{2s}{D}\,. \tag{18}$$

(c) Find the amount of work it takes to move a charge q from the center of the cylinder to its edge.

Solution: Moving a charge q from point $\boldsymbol{a}$ to point $\boldsymbol{b}$ requires work $W = q(V(\boldsymbol{b}) - V(\boldsymbol{a}))$. Therefore the work done from center to edge is just

$$W = q(V(D/2) - V(0)) = -\frac{q\alpha D^3}{8\times 9\epsilon_0}\,. \tag{19}$$

(d) What force (vector) will the charge q experience if it is just outside the edge of the cylinder?

Solution: The electrostatic force on a charge q is $\boldsymbol{F} = q\boldsymbol{E}$. Evaluating this at $s = \epsilon + D/2$ for some infinitesimal positive ϵ (just outside the cylinder), we get

$$\boldsymbol{F} = \frac{q\alpha D^2}{12\epsilon_0}\hat{\boldsymbol{s}}\,. \tag{20}$$

3. **Charged ball**. Your friend Alice has found that the electric potential due to a specially-prepared charged ball of radius a is

$$V(\vec{\boldsymbol{r}}) = \begin{cases} \dfrac{\beta}{\epsilon_0}a^2 - \dfrac{\beta}{6\epsilon_0}\dfrac{r^3}{a} & r \le a \\ \dfrac{5\beta}{6\epsilon_0}\dfrac{a^3}{r} & r \ge a\,, \end{cases} \tag{21}$$

where β is a positive constant.

(a) What is the electric field $\boldsymbol{E}$ everywhere (inside and outside the ball)?

Solution: The electric field is found from the potential via $\boldsymbol{E} = -\nabla V$. Taking derivatives in the two regions,

$$\boldsymbol{E} = \begin{cases} \dfrac{\beta r^2}{2a\epsilon_0}\hat{\boldsymbol{r}} & r \le a \\ \dfrac{5\beta a^3}{6\epsilon_0 r^2}\hat{\boldsymbol{r}} & r \ge a\,. \end{cases} \tag{22}$$

(b) What is the volume charge density ρ everywhere?

Solution: The volume charge density is related to the electric field via Gauss's law (differential version), $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$. Multiplying by ϵ_0 and taking the divergence of the above electric field,

$$\rho = \begin{cases} 2\beta r/a & r \le a \\ 0 & r \ge a\,. \end{cases} \tag{23}$$

(c) What is the surface charge density on the surface of the ball at $r = a$?

Solution: Across a surface charge, the perpendicular component of the electric field changes according to $E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \sigma/\epsilon_0$. The perpendicular component is simply the radial component. Plugging in at $r = a$ and taking the difference (above–below), and multiplying by ϵ_0,

$$\sigma = \frac{a\beta}{3}\,. \tag{24}$$

(d) Find the total amount of energy stored in the electric field configuration.

Solution: One approach is to use the integral

$$E = \frac{1}{2}\int_{\text{all space}} \rho V\, d^3\text{Vol}\,. \tag{25}$$

We have to remember to include the contribution from the surface charge, so we could rewrite this

$$E = \frac{1}{2}\left[\int_{\text{all space}} \rho V\, d^3V + \int_{\text{all surfaces}} \sigma V\, d^2a\right]\,. \tag{26}$$

Plugging in, we have to compute

$$E = \frac{1}{2}\left[\int_0^a \frac{2\beta r}{a}\left(\frac{\beta}{\epsilon_0}a^2 - \frac{\beta}{6\epsilon_0}\frac{r^3}{a}\right)4\pi r^2\, dr + \frac{a\beta}{3}\frac{5\beta a^2}{6\epsilon_0}4\pi a^2\right]\,, \tag{27}$$

where we used spherical symmetry. This evaluates to

$$E = \frac{92a^5\beta^2\pi}{63\epsilon_0} \,. \tag{28}$$

We can check this result by computing the energy a different way, from the integral

$$E = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 \, d^3V \,. \tag{29}$$

Now we have to do the integral

$$E = \frac{\epsilon_0}{2} \left[\int_0^a \left(\frac{\beta r^2}{2a\epsilon_0} \right) 4\pi r^2 \, dr + \int_a^\infty \left(\frac{5\beta a^3}{6\epsilon_0 r^2} \right) 4\pi r^2 \, dr \right] \,, \tag{30}$$

again using spherical symmetry. Doing all the integration and algebra, we get agreement,

$$E = \frac{92a^5\beta^2\pi}{63\epsilon_0} \,. \tag{31}$$

4. **Induced charge in a conducting sphere**. One of the cases that we saw could be treated by the method of images was a conducting sphere. Suppose we have a conducting hollow spherical shell of inner radius R_1 and thickness h (so the outer radius is $R_2 = R_1 + h$), centered at the origin, with a net zero charge on the conductor. We place a charge q' inside, at coordinates $(0, 0, b)$, $0 \leq b \leq R_1$.

   (a) Where do we imagine an image charge, and what is the value of its charge, in order to find the electric potential V inside (for $0 \leq r \leq R_1$)? Find V in this region.

   Solution: This is like the "reciprocal" of Example 3.2 (in the 3rd Ed. of Griffiths). The role of the real charge and image charge have swapped. The crucial fact, though, is that with two charges, there exist equipotential *spheres* around either charge. So, a conducting sphere at one of those surfaces can be treated with the method of images.

   So, we just take the solution from Ex. 3.2. Now our image charge will be located at $(0, 0, a)$, where $a = R_1^2/b$ (a "reflection through the sphere"), and the value of the charge will be $q = -\frac{a}{R_1} q' = -\frac{R_1}{b} q'$.

   The potential interior to R_1 is, up to an additive constant, the same as that of the real and image charges,

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'} \right) + \text{const.} \,, \tag{32}$$

   where $\mathscr{r}$ and $\mathscr{r}'$ are the distances from q and q' respectively. Notice that when $r = R_1$, the two terms inside the parentheses cancel (Griffiths Prob. 3.7). So, the value of the potential on the surface $r = R_1$ is the value of the additive constant, which we will find below by matching with the exterior solution.

   (b) What are the electric field $\boldsymbol{E}$ and potential V in the region $R_1 \leq r \leq R_2$, and outside, where $R_2 \leq r$?

   Solution: Within the conducting shell, between R_1 and R_2, there can be no electric field (or else charges would be accelerated, and this would not be a static configuration). Thus $\boldsymbol{E} = 0$ and V =const which we will now find by matching to the exterior.

   Recall from the second uniqueness theorem that the total amount of charge on the surface of a conductor is the total amount of charge within any cavities in that conductor. So, there is a total charge q' distributed on the surface $r = R_2$. By symmetry it is distributed uniformly on this surface. So exterior to R_2, the potential and electric field are the same as a point charge q' at the center. Thus we have

$$V(r > R_2) = \frac{1}{4\pi\epsilon_0} \frac{q'}{r} \,, \tag{33}$$

$$\boldsymbol{E}(r > R_2) = \frac{1}{4\pi\epsilon_0} \frac{q'}{r^2} \hat{\boldsymbol{r}} \,. \tag{34}$$

Since the potential is continuous, the value of the constant potential between R_1 and R_2 is the same as this potential at the surface,

$$V(R_1 < r < R_2) = \frac{1}{4\pi\epsilon_0}\frac{q'}{R_2}\,. \tag{35}$$

This is also the constant above in part 4a.

(c) What is the induced surface charge density $\sigma(\vec{r})$ along the inner surface, $r = R_1$? What about on the outer surface $r = R_2$?

Solution: On the outer surface is easy, because by symmetry it will be distributed uniformly. Thus the outer surface charge density is the constant $\sigma_2 = q'/(4\pi R_2^2)$.

On the inner surface we can use the change in the normal derivative of the potential,

$$\hat{\boldsymbol{n}} \cdot (\nabla V_{\text{above}} - \nabla V_{\text{below}}) = -\frac{\sigma}{\epsilon_0}\,. \tag{36}$$

Let's take "above" to mean outside and "below" to mean inside, so $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$. Above, the potential is constant, so we only have to deal with the interior potential. Using the law of cosines, this interior potential is

$$V(r < R_1, \theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{q}{\sqrt{R_1^2 + (ra/R_1)^2 - 2ra\cos\theta}}\right] + \text{const.} \tag{37}$$

We only need the radial derivative to compute σ_1,

$$\hat{\boldsymbol{r}} \cdot \nabla V = \frac{\partial V}{\partial r} = \frac{1}{4\pi\epsilon_0}\left[\frac{-q(r - a\cos\theta)}{(r^2 + a^2 - 2ra\cos\theta)^{3/2}} + \frac{q(a^2 r/R_1^2 - a\cos\theta)}{(R_1^2 + (ra/R_1)^2 - 2ra\cos\theta)^{3/2}}\right]. \tag{38}$$

Evaluating this at $r = R_1$ there is a slight simplification,

$$\left.\frac{\partial V}{\partial r}\right|_{r=R_1} = \frac{1}{4\pi\epsilon_0}\frac{q(a^2 - R_1^2)}{R_1(a^2 + R_1^2 - 2aR_1\cos\theta)^{3/2}}\,. \tag{39}$$

The charge density is then $\sigma_1(\theta) = -\epsilon_0 \partial V/\partial r\big|_{r=R_1}$ using the above expression. You can check that this is correct by integrating over the surface to find that the induced surface charge is $-q'$.

(d) What is the force (vector) on the charge q' due to this induced surface charge?

Solution: The force on q' is the same as the force due to the image charge,

$$\boldsymbol{F} = \hat{\boldsymbol{z}}\frac{1}{4\pi\epsilon_0}\frac{-qq'}{(a-b)^2}\,. \tag{40}$$

(e) What is the total energy of this configuration?

Solution: Let's use the formula

$$E = \frac{1}{2}\int_{\text{all space}} \rho V \, d^3V\,. \tag{41}$$

We have to remember to treat this for the point charge q' and the two surface charges $\sigma_1(\boldsymbol{r})$ and σ_2, so we write

$$E = \frac{1}{2}\left[q' V_{\text{reg}}(0,0,b) + \int_{r=R_1} \sigma_1 V \, d^2a + \int_{r=R_2} \sigma_2 V \, d^2a\right]. \tag{42}$$

Here we are writing V_{reg} for the "regular" part of the potential at the location of the charge q', avoiding its own infinite self-energy. That is, this is the potential due to everything else in the universe besides the charge q'. This value is

$$V_{\text{reg}}(0,0,b) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{b-a} + \frac{q'}{R_2}\right). \tag{43}$$

Now we move on to the two surface integrals. They are both on the surface of a conductor which is an equipotential, so we can take the factors of V out of the integrals. These then just end up being the integrals for the total amount of charge on the inner and outer surfaces, times their potentials. The potentials are the same and the charges are equal but opposite, so these two terms cancel. So the energy is simply

$$E = \frac{q'}{8\pi\epsilon_0}\left(\frac{q}{b-a} + \frac{q'}{R_2}\right) . \tag{44}$$