UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism I (Phys. 401) — Prof. Leo C. Stein — Fall 2019

Problem Set 7 — SOLUTIONS

Due: Monday, Oct. 28, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. **Basis decompositions**. This is practice for decomposing functions into some of the more common bases: series of Legendre polynomials $P_n(x)$, (complex) Fourier series e^{inx}, and spherical harmonics $Y_l^m(\theta,\phi)$. You can perform the decompositions by identifying coefficients, or the more systematic approaching of using orthogonality & completeness of basis functions. The orthogonality relations are:

$$\int_0^{2\pi} e^{-inx}e^{imx}dx = 2\pi\delta_{n,m} \tag{1}$$

$$\int_{-1}^{+1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{n,m} \tag{2}$$

$$\int_0^{2\pi}\int_0^{\pi} \overline{Y_l^m(\theta,\phi)}Y_{l'}^{m'}(\theta,\phi)\sin\theta\,d\theta\,d\phi = \delta_{l,l'}\delta_{m,m'} \tag{3}$$

where n, m, l, l', m' are all integers.

   (a) Decompose $p(u) = 3u^3 - 4u^2 + u$ as a series of Legendre polynomials $P_n(u)$
   Solution: One can of course integrate, but it is in fact easier to match coefficients. Then we find

$$p(u) = \frac{6}{5}P_3(u) - \frac{8}{3}P_2(u) + \frac{14}{5}P_1(u) - \frac{4}{3}P_0(u)\,. \tag{4}$$

   (b) Decompose $q(\theta) = \cos^5(\theta)$ as a series of Legendre polynomials $P_n(u)$
   Solution: With $u = \cos\theta$,

$$q(\theta) = \frac{8}{63}P_5(u) + \frac{4}{9}P_3(u) + \frac{3}{7}P_1(u)\,. \tag{5}$$

   (c) Decompose $r(z) = 1 + \sin^4(z)$ as a complex Fourier series
   Solution: The simplest approach is to use $\sin z = (e^{iz} - e^{-iz})/2i$ and expand. We get

$$r(z) = \frac{11}{8} - \frac{1}{4}\left(e^{2iz} + e^{-2iz}\right) + \frac{1}{16}\left(e^{4iz} + e^{-4iz}\right)\,. \tag{6}$$

   (d) Decompose $w(\theta,\phi) = 4\cos\theta\sin^2\theta\sin(2\phi)$ into spherical harmonics $Y_l^m(\theta,\phi)$
   Solution: By counting the number of azimuthal nodes (i.e. the dependence in ϕ) we see that we can only have $m = 2$ and $m = -2$ appearing. By counting the total number of nodes (azimuthal and polar) we see that we only need $l = 3$. So, let's inspect:

$$Y_3^{\pm 2}(\theta,\phi) = \sqrt{\frac{105}{32\pi}}\cos\theta\sin^2\theta e^{\pm 2i\phi}\,. \tag{7}$$

   So, matching coefficients, we get

$$w(\theta,\phi) = \sqrt{\frac{128\pi}{105}}\left(Y_3^2(\theta,\phi) + Y_3^{-2}(\theta,\phi)\right)\,. \tag{8}$$

2. **Potential inside a cube**. Your friend Brittany has constructed a cubical box with all sides of length L, going from the origin to $x = L$, $y = L$, and $z = L$. Five sides are held at ground, potential $V = 0$. The side in the $x - y$ plane with $z = 0$ is insulated from the other sides and has a constant potential $V = V_0$. What is the potential $V(x, y, z)$ (which depends on all three coordinates) everywhere inside the box?

Solution: We will apply separation of variables in x, y, z. Writing $V(x, y, z) = X(x)Y(y)Z(z)$, the Laplacian acting on V is

$$0 = \nabla^2 V = X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z)\,. \tag{9}$$

Now dividing through by V to separate,

$$0 = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\,. \tag{10}$$

Each term in this sum is a function of only one coordinate but they must sum to zero. Therefore each term must be a constant and the constants sum to zero. We will take the constants to be

$$\frac{X''}{X} = -k_x^2 \tag{11}$$

$$\frac{Y''}{Y} = -k_y^2 \tag{12}$$

$$\frac{Z''}{Z} = +\kappa^2 \tag{13}$$

where $\kappa^2 - \kappa_x^2 - k_y^2 = 0$. We have chosen these signs since the box has two equipotential sides in the x and y directions, so the potential in the x and y directions must oscillate away from 0 and then back to it. The solutions to the above are

$$X(x) = A\sin(xk_x) + B\cos(xk_x) \tag{14}$$

$$Y(y) = C\sin(yk_y) + D\cos(yk_y) \tag{15}$$

$$Z(z) = E\sinh(z\kappa) + F\cosh(z\kappa)\,. \tag{16}$$

Now we begin applying boundary conditions. The boundary conditions at $x = 0$, $x = L$ tell us that $B = 0$ and $k_x = n\pi/L$ for some integer n. Similarly the B.C.s at $y = 0$, $y = L$ tell us that $D = 0$ and $k_y = m\pi/L$ for some integer m. Finally, the z direction is easier to analyze if we shift the argument of $Z(z)$ by L, so that the cosh and sinh are centered around the side held at $V = 0$ instead of $V = V_0$. That is, we can redefine the coefficients to instead write $Z(z) = E'\sinh((z - L)\kappa) + F'\cosh((z - L)\kappa)$. Then the B.C. at $z = L$ tells us that $F' = 0$. So, the form of our potential is now

$$V = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\sin(n\pi x/L)\sin(m\pi y/L)\sinh(\pi(z - L)\sqrt{n^2 + m^2}/L)\,. \tag{17}$$

Now we need to satisfy the last condition,

$$V_0 = V(x, y, 0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\sin(n\pi x/L)\sin(m\pi y/L)\sinh(-\pi\sqrt{n^2 + m^2})\,. \tag{18}$$

Integrate this against $\sin(n'\pi x/L)\sin(m'\pi y/L)$. The left side is

$$\int_0^L\int_0^L V_0\sin(n'\pi x/L)\sin(m'\pi y/L)dxdy = \begin{cases} 0, & n' \text{ or } m' \text{ is even} \\ \frac{4L^2V_0}{\pi^2 m'n'}, & \text{both } n' \text{ and } m' \text{ are odd} \end{cases} \tag{19}$$

The other side is only non-vanishing when $n = n'$ and $m = m'$, yielding

$$C_{n,m}\sinh(-\pi\sqrt{n^2 + m^2})\frac{L^2}{4}\,. \tag{20}$$

So, we found the coefficients (when n and m are both odd)

$$C_{n,m} = \frac{1}{\sinh(-\pi\sqrt{n^2+m^2})}\frac{16V_0}{\pi^2 mn} \tag{21}$$

So the potential inside the box is

$$V = \frac{16V_0}{\pi^2}\sum_{n=1,3,5,\dots}\sum_{m=1,3,5,\dots}\frac{1}{mn}\sin(n\pi x/L)\sin(m\pi y/L)\frac{\sinh(\pi(z-L)\sqrt{n^2+m^2}/L)}{\sinh(-\pi\sqrt{n^2+m^2})}\,. \tag{22}$$

3. **Oppositely charged hemispheres**. Claire has prepared two insulating hemispherical shells of radius R, centered on the origin. The top hemisphere (with $z > 0$ or $0 \le \theta < \pi/2$) has uniform surface charge density $+\sigma_0$, while the opposite hemisphere ($z < 0$ or $\pi/2 < \theta \le \pi$) has the opposite uniform surface charge density, $-\sigma_0$. Find the first three *nonvanishing* terms of the Legendre expansion of the potential $V(\boldsymbol{r})$ both inside and outside the shell of charge. Remember that the coefficients A_l, B_l are different inside and out.

Solution: Recall that the Legendre series solution for the potential is

$$V(r,\theta) = \sum_{l=0}^{\infty} P_l(\cos\theta)\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)\,, \tag{23}$$

with different coefficients $A^{\text{in,out}}, B^{\text{in,out}}$ inside and outside of the shells. By regularity at the origin, $B^{\text{in}} = 0$, and by regularity at infinity, $A^{\text{out}} = 0$. By continuity at the surface $r = R$, we find that

$$A_l^{\text{in}} R^l = \frac{B_l^{\text{out}}}{R^{l+1}}\,. \tag{24}$$

Now, if we had the surface charge as a Legendre series,

$$\sigma(\theta) = \sum_{l=0}^{\infty}\sigma_l P_l(\cos\theta)\,, \tag{25}$$

then the boundary condition on the jump in the normal derivative of the potential,

$$\frac{\partial V^{\text{out}}}{\partial r} - \frac{\partial V^{\text{in}}}{\partial r} = -\frac{\sigma}{\epsilon_0} \tag{26}$$

would tell us the coefficients,

$$\frac{-\sigma_\ell}{\epsilon_0} = -(\ell+1)\frac{B_\ell^{\text{out}}}{R^{\ell+2}} - \ell A_\ell^{\text{in}} R^{\ell-1}\,. \tag{27}$$

Thus we would be able to solve for

$$A_l^{\text{in}} = \frac{1}{(2l+1)R^{l-1}}\frac{\sigma_l}{\epsilon_0}\,, \qquad B_l^{\text{out}} = \frac{R^{l+2}}{2l+1}\frac{\sigma_l}{\epsilon_0}\,. \tag{28}$$

So, the first three nonvanishing σ_l determine the first three nonvanishing terms in the Legendre series for V. But what are they? Let's go back to Eq. (25), and integrate against some $P_n(\cos\theta)$, and use orthogonality of the Legendre polynomials,

$$\sigma_n = \frac{2n+1}{2}\int_0^{\pi}\sigma(\theta)P_n(\cos\theta)\sin\theta\,d\theta\,. \tag{29}$$

So, we have to perform these integrals for the σ we were handed,

$$\sigma(\theta) = \begin{cases} +\sigma_0\,, & 0 \le \theta < \pi/2\,, \\ -\sigma_0\,, & \pi/2 < \theta \le \pi\,. \end{cases} \tag{30}$$

This is easier to analyze in terms of $u = \cos\theta$, where $\sigma(u)$ is an odd function. Because it is an odd function, only odd P_l's can integrate to nonvanishing σ_l. So we need the integrals σ_1, σ_3, and σ_5. Because we have the product of two odd functions, each integral from $u \in [-1, +1]$ is twice the integral from $u \in [0, +1]$. So we compute:

$$\sigma_1 = \frac{2 \times 1 + 2}{2} 2\sigma_0 \int_0^1 P_1(u)\, du = \frac{3\sigma_0}{2} \tag{31}$$

$$\sigma_3 = \frac{2 \times 3 + 2}{2} 2\sigma_0 \int_0^1 P_3(u)\, du = \frac{-7\sigma_0}{8} \tag{32}$$

$$\sigma_5 = \frac{2 \times 5 + 2}{2} 2\sigma_0 \int_0^1 P_5(u)\, du = \frac{11\sigma_0}{16}\,. \tag{33}$$

These coefficients then go into the A's and B's.

4. **Continuing the multipole expansion**. We've already seen in lecture that the multipole expansion can be expressed either in terms of a Legendre expansion,

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int (r')^{\ell} P_\ell(\cos\gamma) \rho(\boldsymbol{r}')\, d^3\mathrm{Vol}'\,, \tag{34}$$

where $\cos\gamma = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}'$; but we've also seen that we can write the first three term as

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{Q}{r} + \frac{p_i \hat{r}^i}{r^2} + \frac{1}{2} \frac{Q_{ij} \hat{r}^i \hat{r}^j}{r^3} + \dots \right\}, \tag{35}$$

where, by expanding the Legendre polynomials, we've defined the scalar Q, vector p_i, and a rank-2 tensor Q_{ij},

$$Q = \int \rho(\boldsymbol{r}')\, d^3\mathrm{Vol}' \tag{36}$$

$$p_i = \int r'_i \rho(\boldsymbol{r}')\, d^3\mathrm{Vol}' \tag{37}$$

$$Q_{ij} = \int [3 r'_i r'_j - (r')^2 \delta_{ij}] \rho(\boldsymbol{r}')\, d^3\mathrm{Vol}'\,. \tag{38}$$

(a) Find the next term in the series of Eq. (35) by defining an appropriate rank-3 tensor O_{ijk}. Give both your definition for O_{ijk} and how this tensor appears in the expansion of $V(\boldsymbol{r})$ (this way you can get credit if e.g. you include a factor of 2 in one place but compensate with a factor of $\frac{1}{2}$ in another place).

Solution: We need to expand the Legendre polynomial in the $\ell = 3$ term in Eq. (35), using

$$P_3(u) = \frac{5}{2} u^3 - \frac{3}{2} u\,. \tag{39}$$

We will use this in the integral

$$O = \int (r')^3 P_3(\cos\gamma) \rho(\boldsymbol{r}')\, d^3V' \tag{40}$$

$$= \int (r')^3 \left[\frac{5}{2} \cos^3\gamma - \frac{3}{2} \cos\gamma \right] \rho(\boldsymbol{r}')\, d^3V'\,. \tag{41}$$

Now put in $\cos\gamma = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}'$. We want to be able to combine all three factors of the scalar r' with the direction vectors $\hat{\boldsymbol{r}}'$. This is clearly possible in the u^3 term, but the u^1 term needs some help. We can multiply that term by $1 = \hat{\boldsymbol{r}}' \cdot \hat{\boldsymbol{r}}'$ in order to combine all the terms. We get

$$O = \int \left[\frac{5}{2} (\boldsymbol{r}' \cdot \hat{\boldsymbol{r}})^3 - \frac{3}{2} (\boldsymbol{r}' \cdot \hat{\boldsymbol{r}})(\boldsymbol{r}' \cdot \boldsymbol{r}') \right] \rho(\boldsymbol{r}')\, d^3V'\,. \tag{42}$$

Next, we want to be able to pull out a factor of $\hat{r}^i\hat{r}^j\hat{r}^k$. There are three factors of $\hat{\boldsymbol{r}}$ in the first term but again the second term will need an insertion of $1 = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}} = \hat{r}^j\hat{r}^k\delta_{jk}$. Now we have

$$O = \hat{r}^i\hat{r}^j\hat{r}^k O_{ijk} \tag{43}$$

where we have defined the tensor integral

$$O_{ijk} = \int \left[\frac{5}{2}r'_i r'_j r'_k - \frac{3}{2}(r')^2 r'_i \delta_{jk}\right] \rho(\boldsymbol{r}')\, d^3V' \,. \tag{44}$$

This tensor looks like it treats the i index differently from the j, k indices. Since we only wrote it being contracted with $\hat{r}^i\hat{r}^j\hat{r}^k$, that index is not actually being treated differently, but people like to make it more manifestly symmetric. Therefore people usually define the octupole tensor as something like

$$O_{ijk} = \int \left[\frac{5}{2}r'_i r'_j r'_k - \frac{1}{2}\left((r')^2 r'_i \delta_{jk} + (r')^2 r'_j \delta_{ki} + (r')^2 r'_k \delta_{ij}\right)\right] \rho(\boldsymbol{r}')\, d^3V' \,. \tag{45}$$

Putting it together we would have the combination

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{Q}{r} + \frac{p_i\hat{r}^i}{r^2} + \frac{1}{2}\frac{Q_{ij}\hat{r}^i\hat{r}^j}{r^3} + \frac{1}{3!}\frac{(3!O_{ijk})\hat{r}^i\hat{r}^j\hat{r}^k}{r^4} + \dots\right\}, \tag{46}$$

where I have just multiplied and divided by a factor of 3! to make the prefactors make sense, so potentially the combination $3!O_{ijk}$ is what others would call the octupole tensor. Again, conventions vary.

(b) Suppose we arrange charges at the four corners of a square in rectangular coordinates at $(\pm d, \pm d, 0)$. The charges in the $++$ and $--$ positions are $+q$, while the charges in the $+-$ and $-+$ positions are $-q$. For this charge configuration, evaluate all the components of these tensors: Q, p_i, Q_{ij}. Evaluate the components of O_{ijk} if you have some free time on your hands.

Solution: The total charge is simply $Q = q - q + q - q = 0$.

The dipole is (with $I = 1, 2, 3, 4$ labeling the 4 particles)

$$\boldsymbol{p} = \sum_I q_I \boldsymbol{r}_I = q(d, d, 0) - q(d, -d, 0) + q(-d, -d, 0) - q(-d, d, 0) \tag{47}$$

$$\boldsymbol{p} = \boldsymbol{0} \,. \tag{48}$$

Finally, evaluating the quadrupole, we need to find $Q_{xx}, Q_{xy}, Q_{xz}, Q_{yy}, Q_{yz}$ where

$$Q_{ij} = \sum_I q_I[3r_i^I r_j^I - (r^I)^2\delta_{ij}] \tag{49}$$

Doing all the sums, we get the 3×3 matrix

$$Q_{ij} = \begin{pmatrix} 0 & 12d^2q & 0 \\ 12d^2q & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{50}$$