UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism I (Phys. 401) — Prof. Leo C. Stein — Fall 2019

Problem Set 11 — SOLUTIONS

Due: Friday, Dec. 13, 2019, by 5PM

Material: The final covers the material so far (up through and including Griffiths' chapter 6).

Due date: Friday, Dec. 13, 2019 by 5PM to 205 Lewis Hall. If my door is closed, please slide the exam under my door. Late exams will require extenuating circumstances.

Logistics: The exam consists of this page plus two page of questions. Do not look at the problems until you are ready to start it.

Time: The work might expand to eat up as much time as you allot – therefore I highly recommend you restrict yourself to no more than 5 hours cumulative time spent on these problems. You may take as many breaks as you like, not counted against the 5 hours. **You should not be consulting references, working on the problems, or discussing with others during the breaks**.

Resources: The midterm and final are **not collaborative**. All questions must be done on your own, without consulting anyone else. You may consult your own notes (both in-class and notes on this class you or a colleague in the class have made), the textbook by Griffiths, and solution sets on the course website. **You may not consult any other material**, including other textbooks, the web (except for the current Phys. 401 website), material from previous years' Phys. 401 or any other classes, or copies you have made of such material, or any other resources. Calculators and symbolic manipulation programs are not allowed.

1. (a) Is there any charge density ρ that generates the electric field

$$\boldsymbol{E} = \frac{\alpha}{4\pi} \frac{r^2 - 1}{(1 + r^2)^2} \hat{\boldsymbol{r}} \,, \tag{1}$$

where α is some constant? If no, why not? If yes, what is that ρ?

Solution: Curl of this electric field vanishes, so yes. Using $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$, we get

$$\rho = \frac{\alpha\epsilon_0}{2\pi} \frac{3r^2 - 1}{r(1 + r^2)^3} \,. \tag{2}$$

(b) Is there any current density $\boldsymbol{J}$ that can generate the magnetic field

$$\boldsymbol{B} = \alpha \frac{2 + s}{(1 + s)^2} \hat{\boldsymbol{z}} \,, \tag{3}$$

where α is a constant (and as usual $s^2 = x^2 + y^2$ in cylindrical coordinates). If no, why not? If yes, what is that $\boldsymbol{J}$?

Solution: Divergence of this magnetic field vanishes, so yes. Using $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$, we get

$$\boldsymbol{J} = \frac{\alpha}{\mu_0} \frac{3 + s}{(1 + s)^3} \hat{\boldsymbol{\phi}} \,. \tag{4}$$

2. Suppose there is an infinite straight wire lying along the $+\hat{\boldsymbol{z}}$ axis. We place a charge density λ along this wire, and force those charges to move in the $+\hat{\boldsymbol{z}}$ direction with a steady velocity v.

(a) What are the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ created by these charges?

Solution: The electric field due to an infinite line charge is (See Ex. 2.1)

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{s} \hat{\boldsymbol{s}} \,. \tag{5}$$

The current is $\boldsymbol{I} = \lambda v \hat{\boldsymbol{z}}$. The magnetic field can be found by applying the integral form of Ampere's law, giving

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}} \,. \tag{6}$$

(b) Now we transport a charge Q from distance b away from the wire to distance a away from the wire. How much work was done on the charge?

Solution: Magnetic fields do no work, so we can simply ignore them for this part. The work that is done can be found from the potential,

$$W = Q(V(a) - V(b)) \,. \tag{7}$$

So, we need to find a potential that gives the above electric field. Up to an additive constant, the potential is

$$V = -\frac{2\lambda}{4\pi\epsilon_0} \ln s + \text{const.} \tag{8}$$

(As an aside, here we are taking the log of a quantity with dimensions of length, and that makes no sense. But we can always do something like $\ln(s/(1\text{ cm}))$ to make it ok, and changing the arbitrary constant can be absorbed into the arbitrary additive constant). So, the amount of work is

$$W = \frac{-2Q\lambda}{4\pi\epsilon_0} \ln \frac{a}{b} \,. \tag{9}$$

(c) Suppose we give this charge, still at distance a, a velocity $w\hat{\boldsymbol{s}}$ directed away from the wire. What is the total electromagnetic force on this charge?

Solution: We use the Lorentz force, $\boldsymbol{F} = Q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$. The direction of the cross product we need is $\hat{\boldsymbol{s}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{z}}$. So we find the force

$$\boldsymbol{F} = Q\left(\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{a}\hat{\boldsymbol{s}} + \frac{w\mu_0 I}{2\pi a}\hat{\boldsymbol{z}}\right) \tag{10}$$

(d) Now suppose we take a magnetic dipole $\boldsymbol{m} = m\hat{\boldsymbol{s}}$ pointing away from the wire. We transport this dipole from distance b away from the wire to distance a away from the wire. How much work was done on the dipole?

Solution: The energy of a magnetic dipole in a magnetic field is $U = -\boldsymbol{m}\cdot\boldsymbol{B}$ (see Prob. 6.21). So we take the energy difference and find that the work is

$$W = -\boldsymbol{m}\cdot(\boldsymbol{B}(a) - \boldsymbol{B}(b)) . \tag{11}$$

However note that the $\boldsymbol{B}$ field at all distances points in the $\hat{\boldsymbol{\phi}}$ direction, and thus the work vanishes, $W = 0$.

(e) What is the torque on this dipole?

Solution: The torque on a magnetic dipole in a magnetic field is $\boldsymbol{N} = \boldsymbol{m} \times \boldsymbol{B}$. The cross product we need is $\hat{\boldsymbol{s}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{z}}$. So, the torque is

$$\boldsymbol{N} = \frac{m\mu_0 I}{2\pi a}\hat{\boldsymbol{z}} . \tag{12}$$

(f) Suppose the dipole has now been rotated (perhaps by the just-computed torque, or some other reason) so that it points in the same direction as the vector $\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{s}}$. What is the force on this dipole?

Solution: Let's start by getting the normalization of the dipole right. The unit vector in the direction of $\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{s}}$ is $\hat{\boldsymbol{n}} = (\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{s}})/\sqrt{2}$.

Now the force on a magnetic dipole in a magnetic field is $\boldsymbol{F} = \nabla(\boldsymbol{m}\cdot\boldsymbol{B})$. The $\boldsymbol{B}$ field is purely in the $\hat{\boldsymbol{\phi}}$ direction so only $m_\phi = m/\sqrt{2}$ contributes. We need to evaluate the gradient

$$\boldsymbol{F} = \nabla\left(\frac{m}{\sqrt{2}}\frac{\mu_0 I}{2\pi s}\right) = -\frac{m}{\sqrt{2}}\frac{\mu_0 I}{2\pi s^2}\hat{\boldsymbol{s}} . \tag{13}$$

3. We have placed an insulating shell of radius R centered at the origin. On the surface of this shell, we have distributed charge according to the azimuthally-symmetric surface charge density:

$$\sigma(\theta) = \sigma_0 + \sigma_1\cos\theta + \sigma_2\cos^2\theta , \tag{14}$$

where $\sigma_{0,1,2}$ are constants.

(a) Rewrite σ in terms of a series of Legendre polynomials

Solution: Since the highest power of $\cos\theta$ here is $\cos^2\theta$, we only need $P_0 = 1$, $P_1(\cos\theta) = \cos\theta$, and $P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$. So, we write

$$\sigma(\theta) = \sigma_0' P_0(\cos\theta) + \sigma_1' P_1(\cos\theta) + \sigma_2' P_2(\cos\theta) \tag{15}$$

where we find the primed coefficients σ_i' in terms of the unprimed ones by matching coefficients. Doing so we find

$$\sigma_2' = \frac{2\sigma_2}{3} \tag{16}$$

$$\sigma_1' = \sigma_1 \tag{17}$$

$$\sigma_0' = \sigma_0 + \frac{\sigma_2}{3} \tag{18}$$

(b) What happens to the electric potential $V(r,\theta)$ going across this surface charge? (Give an equation in terms of $\sigma_{0,1,2}$).

Solution: The potential is continuous everywhere, but the first derivative is discontinuous across a sheet of charge. The discontinuity is given by

$$\hat{\boldsymbol{n}} \cdot (\nabla V_{\text{above}} - \nabla V_{\text{below}}) = -\frac{\sigma}{\epsilon_0}\,, \tag{19}$$

where $\hat{\boldsymbol{n}}$ is the unit normal in the direction point "up" along the surface. In this case, the normal is radial, so this turns into

$$\frac{\partial V_{\text{above}}}{\partial r} - \frac{\partial V_{\text{below}}}{\partial r} = -\frac{\sigma}{\epsilon_0}\,. \tag{20}$$

This is true at every angle θ along the surface.

(c) Find the potential $V(r,\theta)$ inside and outside this shell (again in terms of $\sigma_{0,1,2}$).

Solution: This follows from the multipolar decomposition that we went over in class. Recall that the general solution to the Laplace equation $\nabla^2 V = 0$ in spherical coordinates for an azimuthally-symmetric potential is

$$V = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)\,. \tag{21}$$

We will also need the radial derivative,

$$\frac{\partial V}{\partial r} = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(\ell A_\ell r^{\ell-1} - (\ell+1)\frac{B_\ell}{r^{\ell+2}}\right)\,. \tag{22}$$

There will be a different set of coefficients $A_\ell^{\text{in,out}}, B_\ell^{\text{in,out}}$ for inside and outside, and we have to satisfy boundary conditions at $r = R$. Inside, by regularity, all the $B^{\text{in}} = 0$ vanish. Outside, again by regularity (and the vanishing of the potential at infinity), all the $A^{\text{out}} = 0$ vanish.

Now we have to use the boundary conditions, that $V^{\text{in}} = V^{\text{out}}$ and the change in the radial derivative from the previous part. We actually did this in lecture for an arbitrary charge distribution on sphere. Now let's do it for the simpler case of azimuthal symmetry. Setting the two potential equal at the surface and equation coefficients of P_ℓ, we get

$$A_\ell^{\text{in}} R^\ell = B_\ell^{\text{out}} \frac{1}{R^{\ell+1}}\,, \tag{23}$$

$$A_\ell^{\text{in}} = B_\ell^{\text{out}} \frac{1}{R^{2\ell+1}}\,. \tag{24}$$

Next, the derivative condition. Supposing that the charge density is $\sigma = \sum_\ell \sigma'_\ell P_\ell(\cos\theta)$ and again matching coefficients of P_ℓ,

$$\frac{-\sigma_\ell}{\epsilon_0} = -(\ell+1)\frac{B_\ell^{\text{out}}}{R^{\ell+2}} - \ell A_\ell^{\text{in}} R^{\ell-1}\,. \tag{25}$$

Combining the two we get

$$\frac{\sigma_\ell}{\epsilon_0} = (2\ell+1)\frac{B_\ell^{\text{out}}}{R^{\ell+2}}\,. \tag{26}$$

So, plugging in our values for σ'_ℓ in terms of σ_ℓ, our potential coefficients are

$$B_0^{\text{out}} = \frac{(\sigma_0 + \frac{\sigma_2}{3})R^2}{\epsilon_0}\,, \qquad B_1^{\text{out}} = \frac{\sigma_1 R^3}{3\epsilon_0}\,, \qquad B_2^{\text{out}} = \frac{2\sigma_2 R^4}{15\epsilon_0}\,, \tag{27}$$

$$A_0^{\text{in}} = \frac{(\sigma_0 + \frac{\sigma_2}{3})R}{\epsilon_0}\,, \qquad A_1^{\text{in}} = \frac{\sigma_1}{3\epsilon_0}\,, \qquad A_2^{\text{in}} = \frac{2\sigma_2}{15\epsilon_0 R}\,. \tag{28}$$

4. Show how the Laplace equation can be solved in **cylindrical** coordinates using separation of variables. Show the *ordinary* differential equations (ODE) that result, and state all the conditions on the *separation constants*. You should be able to solve all but one ODE in terms of elementary functions (the last one is solved by a special function we have not yet encountered).

   Solution: Laplace's equation in cylindrical coordinates reads

$$0 = \nabla^2 V = \frac{\partial^2 V}{\partial z^2} + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) . \tag{29}$$

   To separate variables, we will pose the ansatz $V = Z(z)\Phi(\phi)S(s)$. Plugging this in to the Laplace equation and everywhere dividing through by V, we get

$$0 = \underbrace{\frac{Z''(z)}{Z}}_{\text{only } z} + \underbrace{\frac{1}{s^2}\frac{\Phi''(\phi)}{\Phi} + \frac{1}{sS(s)}\frac{d}{ds}\left(sS'(s)\right)}_{\text{no } z \text{ dep.}} . \tag{30}$$

   Since the first term depends only on z while the latter two are independent of z, they must both be constants that add to zero. Therefore we get the two equations

$$+\kappa = \frac{Z''(z)}{Z} , \tag{31}$$

$$-\kappa = \frac{1}{s^2}\frac{\Phi''(\phi)}{\Phi} + \frac{1}{sS(s)}\frac{d}{ds}\left(sS'(s)\right) . \tag{32}$$

   First let's handle Eq. (31), which rearranges to $Z''(z) = \kappa Z(z)$ This has solutions $Z = Ae^{z\sqrt{\kappa}} + Be^{-z\sqrt{\kappa}}$. Here if $\kappa > 0$, we have exponentially growing/decaying solutions, whereas if $\kappa < 0$, we will have oscillatory solutions that can be written in terms of $\sin(z\sqrt{|\kappa|})$ and $\cos(z\sqrt{|\kappa|})$ if so desired.

   Now moving on to Eq. (32). Let's rearrange this,

$$0 = \underbrace{\frac{\Phi''(\phi)}{\Phi}}_{\text{only } \phi} + \underbrace{\frac{s}{S(s)}\frac{d}{ds}\left(sS'(s)\right) + \kappa s^2}_{\text{only } s} . \tag{33}$$

   We can again separate, since the two terms have to cancel, they must be opposite constants,

$$-m^2 = \frac{\Phi''(\phi)}{\Phi} \tag{34}$$

$$+m^2 = \frac{s}{S(s)}\frac{d}{ds}\left(sS'(s)\right) + \kappa s^2 . \tag{35}$$

   We must have the negative sign in the ϕ equation since $\phi = 0$ and $\phi = 2\pi$ describe the same point. This also demands that $m \in \mathbb{Z}$. The solutions are $\phi = C\sin(m\phi) + D\cos(m\phi)$.

   Finally, rearranging the s equation, we can write it as

$$0 = s^2 S'' + sS' + (s^2\kappa - m^2)S . \tag{36}$$

   This is known as the Bessel differential equation, though it is not yet in its standard form. To put it in standard form, define a new radial coordinate $\lambda = s\sqrt{\kappa}$, and $\frac{d}{ds} = \sqrt{\kappa}\frac{d}{d\lambda}$. Now we get

$$0 = \lambda^2 \frac{d^2}{d\lambda^2}S + \lambda\frac{d}{d\lambda}S + (\lambda^2 - m^2)S . \tag{37}$$

   The solutions are now $S = EJ_m(\lambda) + FY_m(\lambda)$, where J_m and Y_m are called Bessel functions of the first and second kind, of degree m.

5. (a) Take a sphere of radius R, centered on the origin, that has charge Q distributed uniformly throughout. Cut it into two hemispheres, the "North" ($z > 0$) and "South" ($z < 0$). Discard the Northern hemisphere.

Go to a very large distance $r \gg R$ and expand the electric field $\boldsymbol{E}$ as a power series in powers of $1/r^k$, with k being positive integers. Find the first two non-zero terms in this series (to avoid any potential issues of conventions, please state your solution as $\boldsymbol{E} = \ldots$).

Solution: The $1/r$ expansion in the exterior region is exactly the multipole expansion. We will start with the potential, where we know the first two terms in the multipole expansion are:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2} + \mathcal{O}(r^{-3})\right] . \tag{38}$$

where q is the total charge, and $\boldsymbol{p}$ is the electric dipole moment. The total charge for this configuration is $q = Q/2$, and the dipole is

$$\boldsymbol{p} = \int \boldsymbol{r}'\rho(\boldsymbol{r}')\, d^3V . \tag{39}$$

Because of the azimuthal symmetry of the hemisphere, the dipole will be aligned with the symmetry axis $\hat{\boldsymbol{z}}$, so we only have to do one integral for p_z, not three, since $\boldsymbol{p} = p_z\hat{\boldsymbol{z}}$.

We set up this integral in Cartesian coordinates but perform it in spherical. The density is a constant $\rho = 3Q/(4\pi R^3)$ in the region $0 \le r \le R$, $0 \le \phi \le 2\pi$, and $\pi/2 \le \theta \le \pi$. Setting up the integral,

$$p_z = \int_{\mathcal{V}} z'\rho\, dx'\, dy'\, dz' = \int_0^R \int_0^{2\pi} \int_{\pi/2}^{\pi} (r'\cos\theta')\rho\,(r')^2 \sin\theta'\, d\theta' d\phi' dr' \tag{40}$$

$$= \rho\, 2\pi \left(\int_0^R (r')^3\, dr'\right)\left(\int_{\pi/2}^{\pi} \cos\theta' \sin\theta'\, d\theta'\right) \tag{41}$$

$$p_z = \frac{3Q}{2R^3}\left(\frac{R^4}{4}\right)\left(\frac{-1}{2}\right) = \frac{-3RQ}{16} . \tag{42}$$

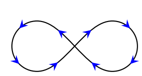
So, the expansion of the potential is

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{2r} + \frac{-3RQ\cos\theta}{16r^2} + \mathcal{O}(r^{-3})\right] . \tag{43}$$

Now all we have to do is take the gradient to get the electric field, $\boldsymbol{E} = -\nabla V$,

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{2r^2}\hat{\boldsymbol{r}} + \frac{-3RQ}{16r^3}\left(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right) + \mathcal{O}(r^{-3})\right] . \tag{44}$$

(b) Suppose we flow current I through a 'figure 8' shaped wire, with the current going like so:



The two halves have the same shape, with each loop having radius R, and the central segments crossing at right angles. The wire lies in the $x - y$ plane.

Go to a very large distance $r \gg R$ and expand the magnetic field $\boldsymbol{B}(r)$ as a power series in powers of $1/r^k$, with k being positive integers. What is the leading term in the series (i.e. lowest k whose coefficient does not vanish) for this current distribution?

Solution: Now we use the multipole expansion of the magnetic vector potential,

$$\boldsymbol{A} = \frac{\mu_0 I}{4\pi}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}\oint (r')^\ell P_\ell(\cos\gamma) d\boldsymbol{l}' . \tag{45}$$

The first term $\ell = 0$ would be a magnetic monopole, but this vanishes because $\oint d\boldsymbol{l}'$ around a closed loop is 0. The second term $\ell = 1$ is a magnetic dipole. We saw a more convenient way to write it in terms of the directed area $\boldsymbol{a} = \int d^2\boldsymbol{a}$ as

$$\boldsymbol{A}_{\text{dip}} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m} \times \hat{\boldsymbol{r}}}{r^2} \tag{46}$$

where $\boldsymbol{m} = I\boldsymbol{a}$. Now we have to see if $\boldsymbol{a}$ and thus $\boldsymbol{m}$ is nonzero. We can compute the directed area as the sum of the left and right halves of the figure 8, which have the same shape. However notice that the left half's area vector is directed out of the page, while the right half's is directed into the page. So, the sum of these two areas cancels, and $\boldsymbol{m} = 0$. Therefore, the first nonvanishing moment is going to be $\ell = 2$, which makes $\boldsymbol{B} \propto 1/r^4$.

6. Let's take a very long cylinder of radius R, with its symmetry axis along the z axis. This cylinder is made of a linear magnetic material with magnetic susceptibility χ_m. The magnetic field inside is

$$\boldsymbol{B} = \frac{\alpha\mu_0}{2}(1+\chi_m)s^2\,\hat{\boldsymbol{z}}\,, \tag{47}$$

where α is a constant. Find the following quantities:

(a) The auxiliary field $\boldsymbol{H}$,

Solution: For a linear magnetic medium, the relationship between $\boldsymbol{B}$ and $\boldsymbol{H}$ is $\boldsymbol{B} = \mu\boldsymbol{H}$, where $\mu = \mu_0(1+\chi_m)$. So, we find

$$\boldsymbol{H} = \frac{\alpha}{2}s^2\hat{\boldsymbol{z}}\,. \tag{48}$$

(b) the magnetization $\boldsymbol{M}$,

Solution: The magnetization is related to $\boldsymbol{H}$ via $\boldsymbol{M} = \chi_m\boldsymbol{H}$ in a linear magnetic medium, so we have

$$\boldsymbol{M} = \chi_m\frac{\alpha}{2}s^2\hat{\boldsymbol{z}}\,. \tag{49}$$

(c) the bound volume current density $\boldsymbol{J}_b$ inside the cylinder,

Solution: The bound volume current is $\boldsymbol{J}_b = \nabla \times \boldsymbol{M}$, so we compute

$$\boldsymbol{J}_b = -s\alpha\chi_m\hat{\boldsymbol{\phi}}\,. \tag{50}$$

(d) the bound surface current density $\boldsymbol{K}_b$ on the surface of the cylinder, and

Solution: The bound surface current is $\boldsymbol{K}_b = \boldsymbol{M} \times \hat{\boldsymbol{n}}$. For our cylinder, the surface at $s = R$ has unit normal $\hat{\boldsymbol{n}} = \hat{\boldsymbol{s}}$. Taking the cross product we get

$$\boldsymbol{K}_b = \chi_m\frac{\alpha}{2}R^2\hat{\boldsymbol{\phi}}\,. \tag{51}$$

(e) the magnetic field $\boldsymbol{B}$ at an infinitesimal distance outside of the cylinder.

Solution: Here we use the boundary conditions across a current sheet $\boldsymbol{K}$. There is no free surface current, only the bound surface current, $\boldsymbol{K} = \boldsymbol{K}_b$. So we use the boundary condition

$$\boldsymbol{B}_{\text{above}} - \boldsymbol{B}_{\text{below}} = \mu_0\boldsymbol{K} \times \hat{\boldsymbol{n}}\,. \tag{52}$$

We take the inside to be "below" and the outside "above," so that $\hat{\boldsymbol{n}} = \hat{\boldsymbol{s}}$. Taking the cross product and inserting the value of $\boldsymbol{B}_{\text{below}}$, we get

$$\boldsymbol{B}_{\text{above}} = \mu_0\chi_m\frac{\alpha}{2}R^2(-\hat{\boldsymbol{z}}) + \frac{\alpha\mu_0}{2}(1+\chi_m)R^2\,\hat{\boldsymbol{z}}\,, \tag{53}$$

$$\boldsymbol{B}_{\text{above}} = \frac{\mu_0\alpha R^2}{2}\hat{\boldsymbol{z}}\,. \tag{54}$$