UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2020

Problem Set 1 — SOLUTIONS

Due: Monday, Feb. 3, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. **Practice with index notation**. Remember that we're using the *Einstein summation convention*, which means that when an index is repeated, there is an implicit sum over all the values it takes. For example, if we have two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, they each have three components A_i where $i = 1, 2, 3$ which are usually called $A_1 = A_x$, $A_2 = A_y$, and so on; then their dot product can be written as

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_i B^i = \sum_{i=1}^{3} A_i B^i = A_1 B^1 + A_2 B^2 + A_3 B^3 \,. \tag{1}$$

The basic objects we have to work with are the Kronecker delta, δ_{ij}, which is 1 when $i = j$ and 0 otherwise; the Levi-Civita tensor or alternating or completely antisymmetric tensor ϵ_{ijk} which is $+1$ when $ijk = 1\,2\,3$ or a cyclic permutation ($2\,3\,1$ or $3\,1\,2$), and is -1 when $ijk = 3\,2\,1$ or a cyclic permutation, and is 0 otherwise; and ∇_i which is a derivative operator that can give div, grad, or curl, depending on how it's combined with the above. Examples: $(\nabla f)_i = \nabla_i f$, $\nabla \cdot \boldsymbol{A} = \nabla_i A^i$, while

$$(\nabla \times \boldsymbol{A})_i = \epsilon_{ijk} \nabla_j A_k \,. \tag{2}$$

(a) δ_{ij} is basically the identity matrix. What is $\delta_{ij} A^j$? What is $\delta_i{}^i$ (using the Einstein summation convention)?

Solution: Taking the implied sum over j, the ith component of $\delta_{ij} A^j$ only picks out A_i. That is,

$$\delta_{ij} A^j = A_i \,. \tag{3}$$

This is a more general rule – whenever one index (say the second) of δ_{ij} is repeated and "contracted" onto another vector or tensor, you get to replace that vector's/tensor's contracted index with the other (say the first) index of δ_{ij}.

When you perform the implied sum in $\delta_i{}^i$, every term is 1 and you have d terms where d is the spatial dimension. Thus $\delta_i{}^i = d$ which is 3 for us.

(b) Similar to what we did in class, show the identity $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$ for a vector field $\boldsymbol{A}$, in index notation.

Solution: Converting the divergence and cross products to index notation, we have

$$\nabla \cdot (\nabla \times \boldsymbol{A}) = \nabla_i (\nabla \times \boldsymbol{A})^i = \nabla_i \epsilon^{ijk} \nabla_j A_k = \epsilon^{ijk} \nabla_i \nabla_j A_k \,. \tag{4}$$

Now using the fact that partial derivatives commute,

$$\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} \quad \Longrightarrow \quad \nabla_i \nabla_j = \nabla_j \nabla_i \,, \tag{5}$$

so we say that $\nabla_i \nabla_j$ is *symmetric* on i and j. The second rule we can use is that the epsilon tensor is *antisymmetric* under exchange of any two indices,

$$\epsilon_{ijk} = -\epsilon_{ikj} \,. \tag{6}$$

Now whenever two symmetric indices are contracted onto two antisymmetric indices, the term must vanish. The way to prove this is always the same: rename dummy indices, then commute them and show that an expression must be equal to its own negative:

$$\nabla \cdot (\nabla \times \boldsymbol{A}) = +\epsilon^{ijk} \nabla_i \nabla_j A_k \tag{7}$$
$$\nabla \cdot (\nabla \times \boldsymbol{A}) = +\epsilon^{jik} \nabla_j \nabla_i A_k \qquad \text{(rename dummies)} \tag{8}$$
$$\nabla \cdot (\nabla \times \boldsymbol{A}) = -\epsilon^{ijk} \nabla_j \nabla_i A_k \qquad \text{(antisymmetry of } \epsilon) \tag{9}$$
$$\nabla \cdot (\nabla \times \boldsymbol{A}) = -\epsilon^{ijk} \nabla_i \nabla_j A_k \qquad \text{(symmetry of } \nabla_j \nabla_k) \tag{10}$$
$$\nabla \cdot (\nabla \times \boldsymbol{A}) = -\nabla \cdot (\nabla \times \boldsymbol{A}) = 0\,. \tag{11}$$

(c) Using the identity

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}\,, \tag{12}$$

expand $[\nabla \times (\nabla \times \boldsymbol{A})]_i$ in terms of div, grad, and the Laplacian $\nabla^2 = \nabla_i \nabla^i$. You will have to do a bit of index renaming and shuffling around!

Note: It's not always easy to remember the identity in Eq. (12). My favorite way is to think of it as a special case of the more general rule using the determinant:

$$\epsilon_{ijk} \epsilon^{abc} = \begin{vmatrix} \delta_i^a & \delta_i^b & \delta_i^c \\ \delta_j^a & \delta_j^b & \delta_j^c \\ \delta_k^a & \delta_k^b & \delta_k^c \end{vmatrix}\,. \tag{13}$$

Solution: The i component of the vector $\boldsymbol{v} = \nabla \times (\nabla \times \boldsymbol{A})$ is written in index notation as

$$v_i = \epsilon_{ijk} \nabla_j \epsilon_{klm} \nabla_l A_m \tag{14}$$
$$v_i = \epsilon_{ijk} \epsilon_{klm} \nabla_j \nabla_l A_m \tag{15}$$

since the ϵ's are just a bunch of 1's and 0's, so they can be pulled through the derivatives.

To apply Eq. (12) we have to permute indices so that it is the first index of each ϵ that is repeated, and then rename indices. Doing so gives

$$v_i = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \nabla_j \nabla_l A_m\,. \tag{16}$$

Now expand the product and use the rule from item 1a to contract the δ's,

$$v_i = \delta_{il} \delta_{jm} \nabla_j \nabla_l A_m - \delta_{im} \delta_{jl} \nabla_j \nabla_l A_m \tag{17}$$
$$v_i = \nabla_j \nabla_i A_j - \nabla_j \nabla_j A_i \tag{18}$$
$$v_i = \nabla_i \nabla_j A_j - \nabla_j \nabla_j A_i\,, \tag{19}$$

where the last step is from the symmetry of partial derivatives. If we rewrite this in terms of traditional vector notation we see the identity

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla (\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}\,. \tag{20}$$

(d) With the "position vector" $\boldsymbol{r}$ that has components $r_1 = x, r_2 = y$, etc., find an index notation expression for

$$\nabla_i r_j \tag{21}$$

Solution: The x derivative of x is 1, but the x derivative of y or z is 0. Similarly if we make a matrix of all 9 possible derivatives of coordinates we see:

$$\nabla_i r_j = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial x}{\partial y} & \frac{\partial y}{\partial y} & \frac{\partial z}{\partial y} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta_{ij}\,. \tag{22}$$

2. Griffiths problem 7.7 (metal bar sliding across rails in a magnetic field).

   Solution:

   (a) $\Phi = Blx$, where x is the horizontal position of the bar, thus $\mathcal{E} = -\frac{d\Phi}{dt} = -Blv$. The voltage across the resistor comes from this EMF, and from $V = IR$ we get that $I = Blv/R$, the current flowing downward (the direction which would create a $\boldsymbol{B}$ field that counters the change in flux through the loop).

   (b) The same current I is flowing upward through the bar, which gives rise to a Lorentz force on each charge carrier, $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$. Instead of discrete charges think of the current $\boldsymbol{J} = \rho\boldsymbol{v}$ which gives rise to the force density $\boldsymbol{f} = \boldsymbol{J} \times \boldsymbol{B}$. Here we take this as a linear force density per unit length of the bar and integrate against its length (just multiply as it is the same quantity at each point). The total force is $\boldsymbol{F} = l(-\hat{\boldsymbol{x}})IB$ or $\boldsymbol{F} = -\boldsymbol{x}B^2\hat{l}^2 v/R$ (the force points left).

   (c) This gives the differential equation $m\ddot{x} = -\alpha\dot{x}$, where the coefficient $\alpha \equiv B^2l^2/R$. In terms of $v = \dot{x}$, this is $\dot{v} = -\alpha v/m$. The solution is $v(t) = v_0 \exp(-\alpha t/m)$.

   (d) The instantaneous power through the resistor is $P = I^2R = B^2l^2v^2/R = \alpha v^2$. We have to integrate this from time $t = 0$ to $t \to \infty$. The square of the exponential in item 2c is another exponential, which is straightforward to integrate. Thus find the total energy dissipated in the resistor as

$$\Delta E = \int_0^t \alpha v^2 dt = \alpha \int_0^t v_0^2 e^{-2\alpha t/m} dt = \frac{-mv_0^2}{2}\left[e^{-2\alpha t/m}\right]_0^\infty = \frac{mv_0^2}{2}\,. \tag{23}$$

3. Griffiths problem 7.8 (square loop moved near a current-carrying wire).

   Solution:

   (a) The magnetic field is given by $\boldsymbol{B} = \hat{\phi}\frac{\mu_0 I}{2\pi s}$. Integrate this dotted into the area of the loop which is also in the $\hat{\phi}$ direction, $\Phi = \frac{\mu_0 I}{2\pi} a \int_s^{s+a} \frac{ds}{s} = \frac{\mu_0 a I}{2\pi} \ln\left(\frac{s+a}{s}\right)$.

   (b) $\mathcal{E} = -\frac{d\Phi}{dt}$. The flux Φ is a function of only the distance s of the bottom of the square, so we can use the chain rule, $\Phi = -\frac{d\Phi}{ds}\frac{ds}{dt} = -\frac{v\mu_0 a I}{2\pi}\frac{d}{ds}\ln\left(\frac{s+a}{s}\right)$. Finally, we get $\mathcal{E} = -\frac{v\mu_0 a I}{2\pi}\left(\frac{1}{s+a} - \frac{1}{s}\right)$. If the loop is moved away, flux is decreasing; thus the EMF will try to create a magnetic field in the same direction to lessen the change in flux, so the current will be counterclockwise.

   (c) The flux would be constant, so $\mathcal{E} = 0$.

4. Griffiths problem 7.22 (self-inductance per unit length of solenoid).

   Solution: When current I flows through the solenoid, it generates a field $B = \mu_0 nI$. If we consider an individual winding of the solenoid, there is flux $\Phi_1 = \pi R^2\mu_0 nI$ in this single loop. The total flux is $\Phi = N\Phi_1 = nl\Phi_1 = l\pi R^2\mu_0 n^2 I$, where N is the total number of windings in the length l of solenoid. Compare to the formula for self-inductance $\Phi = LI$. Thus $L = l\pi R^2\mu_0 n^2$, and if we want the inductance per unit length, we just divide by the length l to find $\mathscr{L} = L/l = \pi R^2\mu_0 n^2$.