Problem Set 3 — SOLUTIONS

Due: Monday, Feb. 24, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. Griffiths problem 7.59 (proving Alfven's theorem).

   Solution:

   (a) As in the other problem, for $\boldsymbol{J}$ to remain finite while $\sigma \to \infty$, we need $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \to 0$ at the same speed as $1/\sigma \to 0$. So we start with $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$. Take the curl of both sides, $\nabla \times \boldsymbol{E} = -\nabla \times (\boldsymbol{v} \times \boldsymbol{B})$. But the left hand side can be replaced, by Faraday's law, with $-\partial \boldsymbol{B}/\partial t$, so we have arrived at the desired result, $\partial \boldsymbol{B}/\partial t = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$.

   (b) From $\nabla \cdot \boldsymbol{B} = 0$, integrating over any volume and using the divergence theorem, $\oint \boldsymbol{B} \cdot d\boldsymbol{a} = 0$ over any closed surface. The three surfaces $\mathcal{S}$, $\mathcal{R}$, and $\mathcal{S}'$ together make a closed surface, but the orientation of (direction of area element) is not consistent for the three surfaces. To make the orientation consistent we need a sign difference between $\int_{\mathcal{S}} \boldsymbol{B}(t+dt) \cdot d\boldsymbol{a}$ and $\int_{\mathcal{S}'} \boldsymbol{B}(t+dt) \cdot d\boldsymbol{a}$. That gives the desired equation

$$\int_{\mathcal{S}'} \boldsymbol{B}(t+dt) \cdot d\boldsymbol{a} + \int_{\mathcal{R}} \boldsymbol{B}(t+dt) \cdot d\boldsymbol{a} = \int_{\mathcal{S}} \boldsymbol{B}(t+dt) \cdot d\boldsymbol{a}\,. \tag{1}$$

   Now you can replace $\int_{\mathcal{S}'} \boldsymbol{B}(t+dt) \cdot d\boldsymbol{a}$ in the equation for the flux change,

$$d\Phi = \int_{\mathcal{S}} \boldsymbol{B}(t+dt) \cdot d\boldsymbol{a} - \int_{\mathcal{S}} \boldsymbol{B}(t) \cdot d\boldsymbol{a} - \int_{\mathcal{R}} \boldsymbol{B}(t+dt) \cdot d\boldsymbol{a}\,. \tag{2}$$

   The first two terms are over the same surface, and we can use Taylor's theorem to approximate $\boldsymbol{B}(t+dt) = \boldsymbol{B}(t) + dt\ \partial \boldsymbol{B}/\partial t$, giving

$$d\Phi = dt \int_{\mathcal{S}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{a} - \int_{\mathcal{R}} \boldsymbol{B}(t+dt) \cdot d\boldsymbol{a}\,. \tag{3}$$

   For the last integral, the ribbon is a Cartesian product of the loop $\mathcal{P} \equiv \partial \mathcal{S}$ (the boundary of $\mathcal{S}$) times a short segment generated by $\boldsymbol{v}dt$. Thus we can parameterize each area element along $\mathcal{R}$ with $d\boldsymbol{a} = d\boldsymbol{l} \times \boldsymbol{v}dt$ (this has the same orientation as $\mathcal{S}$ if $\mathcal{P}$ is traversed counter-clockwise as it is drawn in the book). Then we permute the triple product $\boldsymbol{B} \cdot (d\boldsymbol{l} \times \boldsymbol{v})dt = (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l}dt$. Next we use the curl theorem, backwards, to turn this last integral into one over $\mathcal{S}$,

$$d\Phi = dt \left\{ \int_{\mathcal{S}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{a} - \int_{\mathcal{S}} \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) d\boldsymbol{a} \right\}\,. \tag{4}$$

   But if we combine integrands, the total integrand is $\partial \boldsymbol{B}/\partial t - \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$, which from item 1a vanishes identically.

2. **A highly conducting, magnetized plasma – part 2**. Last week, we saw that in a highly conducting plasma, in the limit of $\sigma \to \infty$, we would find the conditions:

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}\,, \tag{5}$$

therefore

$$\boldsymbol{E} \cdot \boldsymbol{B} = 0 \,. \tag{6}$$

Then we decomposed $\boldsymbol{v} = v_\parallel \hat{\boldsymbol{B}} + \boldsymbol{v}_\perp$, and found

$$\boldsymbol{v}_\perp = \boldsymbol{E} \times \boldsymbol{B}/B^2 \,. \tag{7}$$

(a) Starting from Eq. (6), take the time derivative of this result. Plug in for the time derivatives of the electromagnetic fields using Maxwell's equations. You should now be able to solve for $\boldsymbol{v} \cdot \boldsymbol{B}$, allowing you to find the parallel component $v_\parallel$.
Solution: The partial time derivative of Eq. (6) is

$$0 = \boldsymbol{B} \cdot \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{B}}{\partial t} \,. \tag{8}$$

For the time derivatives of the fields we use Maxwell's equations, so we get

$$0 = \boldsymbol{B} \cdot \frac{1}{\mu_0 \epsilon_0} \left[\nabla \times \boldsymbol{B} - \mu_0 \boldsymbol{J} \right] - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{E}) \,. \tag{9}$$

Now recall that $\boldsymbol{J} = \rho \boldsymbol{v} = \rho(v_\parallel \hat{\boldsymbol{B}} + \boldsymbol{v}_\perp)$. Once you insert this you will be able to solve for $v_\parallel$,

$$\rho v_\parallel \boldsymbol{B} \cdot \hat{\boldsymbol{B}} = \frac{1}{\mu_0} \boldsymbol{B} \cdot (\nabla \times \boldsymbol{B}) - \epsilon_0 \boldsymbol{E} \cdot (\nabla \times \boldsymbol{E}) \,, \tag{10}$$

$$v_\parallel = \frac{1}{\rho |B|} \left[\frac{1}{\mu_0} \boldsymbol{B} \cdot (\nabla \times \boldsymbol{B}) - \epsilon_0 \boldsymbol{E} \cdot (\nabla \times \boldsymbol{E}) \right] \,. \tag{11}$$

(b) Finally, sum up by rewriting $\boldsymbol{J}$ purely in terms of *only* the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields and their derivatives.
Solution: We can replace ρ with $\rho = \epsilon_0 \nabla \cdot \boldsymbol{E}$, and this was all that was missing to write $\boldsymbol{J}$ in terms of just the fields:

$$\boldsymbol{J} = \rho \boldsymbol{v} = \rho \left(v_\parallel \hat{\boldsymbol{B}} + \boldsymbol{v}_\perp \right) \tag{12}$$

$$= \frac{\boldsymbol{B}}{B^2} \left[\frac{1}{\mu_0} \boldsymbol{B} \cdot (\nabla \times \boldsymbol{B}) - \epsilon_0 \boldsymbol{E} \cdot (\nabla \times \boldsymbol{E}) \right] + (\epsilon_0 \nabla \cdot \boldsymbol{E}) \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} \,. \tag{13}$$

Thus in the infinite conductivity limit, ρ and $\boldsymbol{v}$ can be solved for in terms of the electromagnetic fields, and eliminated from the equations. The resulting system of equations and physical systems are called *Force-free electrodynamics*.

3. Griffiths problem 8.5a-d (Infinite parallel-plate capacitor's stress tensor, force per unit area, momentum flux, recoil per unit area).

Solution:

(a) There is no magnetic field, and the electric field is $E_z = -\sigma/\epsilon_0$ with other components vanishing. This gives the stress tensor

$$\boldsymbol{T} = \frac{\sigma^2}{2\epsilon_0} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \,. \tag{14}$$

(b) We are interested in the integrand of $\boldsymbol{F} = \oint \boldsymbol{T} \cdot d\boldsymbol{a}$, over a surface that encloses the top plate. Let the surface be a rectangle and infinitesimally thin. The only contribution to the integral is from the bottom face of this surface, where the area element is $d\boldsymbol{a} = -\hat{\boldsymbol{z}} dx\, dy$ (minus because the outward vector points down). This gives $\boldsymbol{f} = -\hat{\boldsymbol{z}} T_{zz} = -\sigma^2/2\epsilon_0 \hat{\boldsymbol{z}}$.

(c) The momentum flux density in direction $\boldsymbol{n}$ is given by $-\boldsymbol{n}\cdot\boldsymbol{T}$. In our case this is again $-T_{zz} = -\sigma^2/2\epsilon_0$.

(d) This amount of momentum flux density is absorbed in each unit area of the plate, thus the force density is $\boldsymbol{f} = -\sigma^2/2\epsilon_0\hat{\boldsymbol{z}}$, as before.

4. Griffiths problem 8.9a-b (Solenoid with a ring outside, energy flux).

Solution:

(a) The EMF in this wire is given by $\mathcal{E} = -d\Phi/dt$, where $\Phi(t) = \pi a^2\,\mu_0 n I_s(t)$; so $\mathcal{E} = -\mu_0 n dI_s/dt$. This circuit is closed by the resistance of the wire, giving us $\mathcal{E} = I_r R$. The current in the wire is then $I_r = \frac{-1}{R}\pi a^2\,\mu_0 n \frac{dI_s}{dt}$.

(b) Just outside the solenoid, there is an azimuthal electric field due to the changing magnetic field. We compute it from $\nabla\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$ which leads to $\oint \boldsymbol{E}\cdot d\boldsymbol{l} = -d\Phi/dt$, from which we find $2\pi a E_\phi = -\pi a^2\,\mu_0 n dI_s/dt$.

Meanwhile, we can find the magnetic field due to the outer ring of wire. Since the radius b of the wire is very large compared to the radius a of the solenoid, we will approximate this with the magnetic field perfectly along the axis of a loop, $\boldsymbol{B} = \hat{\boldsymbol{z}}(\mu_0 I_r/2)b^2/(b^2+z^2)^{3/2}$.

Combine these to get the Poynting flux, $\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}$. We find

$$\boldsymbol{S} = \frac{1}{\mu_0}\left(\frac{-a\mu_0 n}{2}\frac{dI_s}{dt}\right)\left(\frac{\mu_0 I_r}{2}\frac{b^2}{(b^2+z^2)^{3/2}}\right)\hat{\boldsymbol{r}}\,. \tag{15}$$

We integrate this energy flux over all azimuthal angles, giving a factor of $2\pi a$, and then over all values of z. This gives

$$\frac{dE}{dt} = -\frac{\pi a^2 b^2 n\mu_0 I_r}{2}\frac{dI_s}{dt}\int_{-\infty}^{+\infty}\frac{dz}{(b^2+z^2)^{3/2}}\,. \tag{16}$$

This integral can be performed,

$$\int_{-\infty}^{+\infty}\frac{dz}{(b^2+z^2)^{3/2}} = \left.\frac{z}{b^2\sqrt{z^2+b^2}}\right|_{-\infty}^{+\infty} = \frac{2}{b^2}\,. \tag{17}$$

Thus we find

$$\frac{dE}{dt} = -\pi a^2 n\mu_0 I_r\frac{dI_s}{dt} = I_r^2 R\,. \tag{18}$$

5. **Ramping the current in a solenoid**. Suppose we have a solenoid of radius a aligned with the $\hat{z}$ axis with n turns per unit length.

(a) Suppose we turn on the current through the solenoid so that between $t=0$ and $t=\tau$, the current increases linearly, $I(t) = I_1\frac{t}{\tau}$. What is the magnetic field $\boldsymbol{B}$ in the solenoid? From Maxwell's equations, what is the electric field $\boldsymbol{E}$?

Solution: If the current is increasing slowly (so it is in the quasistatic regime, $\tau \gg ca$), the magnetic field will be given by $\boldsymbol{B}(t) = \mu_0 n I(t)\hat{\boldsymbol{z}} = \mu_0 n I_1 t/\tau\hat{\boldsymbol{z}}$. In response to this slowly time-varying magnetic field, an azimuthal electric field will be created. From $\nabla\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$, integrate across a disk of radius s confined to $z = const$. From the curl theorem this gives $\oint\boldsymbol{E}\cdot d\boldsymbol{l} = -d\Phi/dt$ where $\Phi = \int\boldsymbol{B}\cdot d\boldsymbol{a}$ is the flux passing through the disk. This gives us

$$2\pi s E_\phi = -\frac{d}{dt}\pi s^2\mu_0 n I_1\frac{t}{\tau} \tag{19}$$

$$\boldsymbol{E} = -\frac{s\mu_0 n I_1}{2\tau}\hat{\boldsymbol{\phi}}\,. \tag{20}$$

(b) Find the energy flux and the momentum density in the electromagnetic field inside the solenoid.

Solution: The energy flux is given by the Poynting vector,

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B} \tag{21}$$

$$= \frac{1}{\mu_0}\frac{-s\mu_0 n I_1}{2\tau}\mu_0 n I_1 \frac{t}{\tau}\hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{z}} \tag{22}$$

$$\boldsymbol{S} = -\frac{\mu_0 s n^2 I_1^2 t}{2\tau^2}\hat{\boldsymbol{s}}\,. \tag{23}$$

You can see that energy is flowing towards the axis from the edge of the solenoid. The momentum density is $\boldsymbol{\wp} = \mu_0\epsilon_0\boldsymbol{S}$ and so is proportional to the above.

(c) Find the Maxwell stress tensor in the basis of $\hat{z}, \hat{s}, \hat{\phi}$ (i.e. you are looking for components like $T_{\phi\phi}$, T_{zs}, etc.).

Solution: It is very easy to compute E^2 and B^2,

$$E^2 = \left(\frac{s\mu_0 n I_1}{2\tau}\right)^2 \tag{24}$$

$$B^2 = \left(\mu_0 n I_1 \frac{t}{\tau}\right)^2\,. \tag{25}$$

Now examine the form of T_{ij},

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)\,. \tag{26}$$

We can write each term as a matrix in the basis (s, ϕ, z). The electric part is

$$\begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}\epsilon_0 E^2\,, \tag{27}$$

and similarly the magnetic part ic

$$\begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\frac{1}{\mu_0}B^2\,. \tag{28}$$

The full Maxwell stress tensor is the sum of these two terms.