UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
GR II (Phys. 750) — Prof. Leo C. Stein — Spring 2020

Problem Set 1 — SOLUTIONS

Due: Tuesday, Feb. 4, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. Construct explicit examples (giving coordinate maps) of the following types of maps between manifolds. For each of these cases, pick for the domain and codomain any well-known manifold such as the real line $\mathbb{R}$ or space $\mathbb{R}^n$, an interval $[0, 1]$, an n-sphere S^n, Cartesian products of these, etc. (If you are using a non-standard coordinate system then explain the coordinates.)

   (a) injective but not bijective
   Example: The inclusion of the unit radius n-sphere $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$ in $(n+1)$-dimensional space is injective, since everything in the image $\iota(S^n)$ only has one point in the preimage. It is not surjective because only the points a distance 1 from the origin in R^{n+1} are in the image.
   Coordinate construction: The n-sphere has n angle variables. One of them is $\phi \in [0, 1\pi)$ with the identification $0 \sim 2\pi$. The other $n-1$ are named $\theta^{(i)} \in [0, \pi]$, $i = 1 \ldots n-1$. The injection into $\mathbb{R}^{n+1}$ is

$$
\begin{aligned}
x^{(1)} &= \sin\theta^{(n-1)} \sin\theta^{(n-2)} \cdots \sin\theta^{(2)} \sin\theta^{(1)} \cos\phi\,, \\
x^{(2)} &= \sin\theta^{(n-1)} \sin\theta^{(n-2)} \cdots \sin\theta^{(2)} \sin\theta^{(1)} \sin\phi\,, \\
x^{(3)} &= \sin\theta^{(n-1)} \sin\theta^{(n-2)} \cdots \sin\theta^{(2)} \cos\theta^{(1)}\,, \\
x^{(4)} &= \sin\theta^{(n-1)} \sin\theta^{(n-2)} \cdots \cos\theta^{(2)}\,, \\
&\vdots \\
x^{(n)} &= \sin\theta^{(n-1)} \cos\theta^{(n-1)}\,, \\
x^{(n+1)} &= \cos\theta^{(n-1)}\,.
\end{aligned}
$$

   (b) surjective but not bijective
   Example: Wrapping the circle around itself k times where $k \in \mathbb{Z}, k > 1$. This is surjective because every point in the circle is in the image, but it has k points in its preimage.
   Let the circle be the set of points with norm 1 in the complex plane, of the form $e^{i\phi}$, with $\phi \in [0, 2\pi)$ with the identification $0 \sim 2\pi$. Then let the map for integer k be

$$
P_k(e^{i\phi}) = e^{ik\phi}\,. \tag{1}
$$

   (c) neither surjective nor injective; and
   Example: Wrapping the circle onto just part of itself, in the following way. Let the circle be the set of points with norm 1 in the complex plane, of the form $e^{i\phi}$, with $\phi \in [0, 2\pi)$ with the identification $0 \sim 2\pi$. Now let the map be

$$
F(e^{i\phi}) = e^{i\pi\cos(\phi)/2}\,. \tag{2}
$$

   Notice that the image is a closed subset, $\{e^{i\phi} | \phi \in [0, \pi]\} \subset S^1$, not the whole space. All points in the image have two preimages, except for the endpoints, which only have one.

(d) bijective (but not the identity map, that's too easy!)

Example: An irrational map from the real line into a torus, for example $G : \mathbb{R} \to T^2$,

$$G(x) = (e^{i\phi}, e^{i\alpha\phi}) \,, \tag{3}$$

where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is irrational. The line will wind around the 2-torus, never recurring, hitting every point exactly once. The same construction can work for $\mathbb{R} \to T^k$ if you pick k numbers all of whose ratios are irrational. Notice that although G is continuous and G^{-1} exists, G^{-1} is not continuous. So, it is not a homeomorphism (the two spaces have different topologies, there can not be a homeomorphism).

2. Take the map (say it was called $F : M \to N$) you defined in item 1a and compute the differential dF in the coordinates you used above. Use this differential to compute the pullback of any one-form from N back to M.

Example: We will use the case $F : S^2 \hookrightarrow \mathbb{R}^3$. Two angles (θ, ϕ) map into three rectangular coordinates via

$$x^{(1)} = \sin\theta\cos\phi \tag{4}$$
$$x^{(2)} = \sin\theta\sin\phi \tag{5}$$
$$x^{(3)} = \cos\theta \,. \tag{6}$$

In coordinates, the differential is

$$dF = \begin{bmatrix} \frac{\partial x^{(1)}}{\partial\theta} & \frac{\partial x^{(1)}}{\partial\phi} \\ \frac{\partial x^{(2)}}{\partial\theta} & \frac{\partial x^{(2)}}{\partial\phi} \\ \frac{\partial x^{(3)}}{\partial\theta} & \frac{\partial x^{(3)}}{\partial\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{bmatrix} \,. \tag{7}$$

Now let us consider the one-form $\omega = dx^{(3)}$, which has components $\omega_i = (0, 0, 1)$ in the coordinate basis. Its pullback to the 2-sphere is computed via

$$F^*\omega = \omega_i \frac{\partial x^i}{\partial\theta^A} d\theta^A \,, \tag{8}$$

essentially left-multiplying the row vector ω_i into the above matrix. This gives $F^*\omega = -\sin\theta d\theta$.

Or, suppose we consider the one-form $\kappa = d(\frac{1}{2}r^2) = \sum_k x^{(k)}dx^{(k)}$. The coordinate components are $\kappa_i = (x^{(1)}, x^{(2)}, x^{(3)})$. If we restrict it to the image of F, the pullbacks of the coordinate components are $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Now performing the pullback, after the algebraic dust settles,

$$F^*\kappa = \kappa_i \frac{\partial x^i}{\partial\theta^A} d\theta^A = 0 \,. \tag{9}$$

This one-form annihilates the tangent space of the submanifold that is the image of F.

3. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a logarithmic spiral, so that in rectangular coordinates we have $\gamma(t) = (e^t \cos t, e^t \sin t)$.

(a) What is the pullback $\gamma^* r$ of the function $r = \sqrt{x^2 + y^2}$?

Solution: Here we can be cavalier and substitute in the coordinates,

$$\gamma^* r = \sqrt{e^{2t}\cos^2 t + e^{2t}\sin^2} = e^t. \tag{10}$$

(b) Find the matrix representing the differential $d\gamma$ in these coordinates.

Solution:

$$d\gamma = \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix} = \begin{bmatrix} e^t(\cos t - \sin t) \\ e^t(\cos t + \sin t) \end{bmatrix} \tag{11}$$

(c) Use $d\gamma$ to find the tangent vector to the curve.

Solution: We want to push forward the vector ∂_t, whose coordinate components are the column vector (1). Multiplying this on the right, we get simply

$$d\gamma(\partial_t) = e^t \left((\cos t - \sin t)\partial_x + (\cos t + \sin t)\partial_y \right) . \tag{12}$$

(d) Use $d\gamma$ to pull back the one-form dr.

Solution: The function $r = \sqrt{x^2 + y^2}$ has exterior derivative $dr = \frac{1}{r}(xdx + ydy)$, with coordinate components $(dr)_i = (x/r, y/r)$ as a row vector. Evaluating on the image, these components are $(\cos t, \sin t)$. Taking the pullback we get $\gamma^*(dr) = e^t dt$.

Note: This is the same as the exterior derivative of the pullback of the function, found in item 3a. This is because of the naturality of the exterior derivative under pullback. For any map ϕ and differential form ω, we will have $\phi^*(d\omega) = d(\phi^*\omega)$.

(e) Use $d\gamma$ to pull back the two-form $dx \wedge dy$.

Solution: I will omit most of the details. You can find the pullback of each of dx and dy. Each one is proportional to dt. Now, the wedge product is odd for exchanging one-forms. That is, if α, β are both one-forms, $\alpha \wedge \beta = -\beta \wedge \alpha$. Therefore when we compute something proportional to $dt \wedge dt$, if we apply the antisymmetry, $dt \wedge dt = -dt \wedge dt = 0$.

Note: This is a special case of a more general statement: for an n-dimensional vector space, the dimensionality of the space of kth exterior products (wedge products of k vectors) has dimension $\binom{n}{k}$. When $k > n$, this vanishes. So, on an n-dimensional manifold, all k-forms with $k > n$ are 0.