**UNIVERSITY OF MISSISSIPPI**
Department of Physics and Astronomy
GR II (Phys. 750) — Prof. Leo C. Stein — Spring 2020

**Problem Set 2 — SOLUTIONS**

**Due**: Friday, Feb. 14, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. Suppose we have an algebra $\mathcal{A}$, and any two derivations on that algebra, $D_1$ and $D_2$ (recall that a derivation satisfies the Leibniz rule, $D(ab) = D(a)b + aD(b)$). Show that the commutator $[D_1, D_2](a) = D_1 D_2 a - D_2 D_1 a$ is also a derivation.

   ***Solution:*** Act with the commutator on the product $ab$. Expand all the products by applying the Leibniz rule.

$$[D_1, D_2](ab) = D_1 D_2(ab) - D_2 D_1(ab) = D_1 \left(D_2(a)b + aD_2(b)\right) - D_2 \left(D_1(a)b + aD_1(b)\right) \tag{1}$$

$$\begin{aligned} &= (D_1 D_2 a)b + (D_2 a)(D_1 b) + (D_1 a)(D_2 b) + a(D_1 D_2 b) \\ &\quad - (D_2 D_1 a)b - (D_1 a)(D_2 b) - (D_2 a)(D_1 b) - a(D_2 D_1 b)\,. \end{aligned} \tag{2}$$

   Notice that I have always kept the order of the multiplicands, with $a$ on the left and $b$ on the right – this algebra might be noncommutative!

   Now, the one-derivative terms cancel leaving only the two-derivative terms. Collect terms and notice that we again have the Leibniz rule:

$$[D_1, D_2](ab) = (D_1 D_2 a)b + a(D_1 D_2 b) - (D_2 D_1 a)b - a(D_2 D_1 b) \tag{3}$$

$$= ([D_1, D_2]a)b - a([D_1, D_2]b)\,. \tag{4}$$

   Thus in any algebra, a commutator of derivations is again a derivation.

2. Show that every three vector fields $a, b, c \in \mathfrak{X}(\mathcal{M})$ on a manifold $\mathcal{M}$ satisfy the Jacobi identity,

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0\,. \tag{5}$$

   ***Solution:*** You just need to expand everything and cancel terms. Let's compute the "Jacobiator" of $a, b, c$ acting on some scalar function $f$:

$$[a, [b, c]]f + [b, [c, a]]f + [c, [a, b]]f = a[b, c]f - [b, c]af + b[c, a]f - [c, a]bf + c[a, b]f - [a, b]cf \tag{6}$$

$$\begin{aligned} &= (abcf - acbf) - (bcaf - cbaf) + (bcaf - bacf) - (cabf - acbf) \\ &\quad + (cabf - cbaf) - (abcf - bacf) \end{aligned} \tag{7}$$

$$= 0\,, \tag{8}$$

   all terms cancel.

3. Suppose we have a vector bundle $E$ over the base manifold $\mathcal{M}$, and we have a connection (or covariant derivative) $D$ such that the operation $D_v : \Gamma(E) \to \Gamma(E)$ satisfies:

$$D_v(fs + t) = v(f)s + fD_v(s) + D_v(t) \tag{9}$$

$$D_{fv+w}(s) = fD_v(s) + D_w(s)\,, \tag{10}$$

   for scalar function $f \in C^\infty(\mathcal{M})$, vector fields $v, w \in \mathfrak{X}(\mathcal{M})$, and sections $s, t \in \Gamma(E)$.

   Show the following:

(a) If you have this connection $D$ and another connection $D^0$ (also satisfying these rules), that the difference $D - D^0$ is a tensor, in the sense that it does not take a derivative of its argument:

$$D_v(fs) - D^0_v(fs) = f\left(D_v(s) - D^0_v(s)\right) . \tag{11}$$

The fact that it does not take a derivative of $v$ should be clear from the properties of a connection.
***Solution:*** Let's expand how each connection acts on the product of a function and section:

$$\begin{aligned} D_v(fs) - D^0_v(fs) &= v(f)s + fD_v(s) - v(f)s - fD^0_v(s) && (12)\\ &= f\left(D_v(s) - D^0_v(s)\right) . && (13) \end{aligned}$$

Notice that we did not need to invoke any fiducial "parallel" or partial derivative connection $\partial$, or any "Christoffel symbols" to show this. In fact something like Christoffel symbols $\Gamma^a{}_{bc}$ is only available for the tangent bundle, though some people might apply the name to a tensor like $A_a{}^A{}_B$ when acting on a vector bundle.

(b) We define the operation

$$F(v, w)s = D_v D_w s - D_w D_v s - D_{[v,w]} s . \tag{14}$$

Now it is not clear if $v, w$, or $s$ do or don't get differentiated! Show that $F(v, w)$ is a tensor in the sense that it does not take a derivative of $v, w$, or $s$.
***Solution:*** Let's first show this for the $s$ argument, by plugging in some product $fs$:

$$\begin{aligned} F(v, w)(fs) &= D_v D_w(fs) - D_w D_v(fs) - D_{[v,w]}(fs) && (15)\\ &= D_v\left(w(f)s + fD_w s\right) - D_w\left(v(f)s + fD_v s\right) - ([v, w]f)s - fD_{[v,w]}s && (16)\\ &= v(w(f))s + w(f)D_v s + v(f)D_w s + fD_v D_w s\\ &\quad - w(v(f))s - v(f)D_w s - w(f)D_v s - fD_w D_v s\\ &\quad - ([v, w]f)s - fD_{[v,w]}s . && (17) \end{aligned}$$

Now notice that everything cancels except for

$$F(v, w)(fs) = f(D_v D_w s - D_w D_v s - D_{[v,w]}s) = fF(v, w)s . \tag{18}$$

Therefore this operation is linear in the $s$ slot, it did not take a derivative.
For the $v$ and $w$ slots note that by definition, this operation is manifestly antisymmetric in the two slots, $F(v, w)s = -F(w, v)s$. So, if we prove that it is linear in $v$, the proof automatically applies to $w$ and we'd be done. So, let's see what happens when we insert $fv$ in that slot. First we will need a lemma on what is the vector $[fv, w]$. Let it act on some other scalar function $g$:

$$\begin{aligned} [fv, w]g &= fv(w(g)) - w(fv(g)) = fv(w(g)) - w(f)v(g) - fw(v(g)) && (19)\\ &= \left(f[v, w] - w(f)v\right) g . && (20) \end{aligned}$$

So, we have shown that the vector $[fv, w] = f[v, w] + w(f)v$. Now we are ready to evaluate:

$$\begin{aligned} F(fv, w)s &= D_{fv} D_w s - D_w D_{fv} s - D_{[fv,w]} s && (21)\\ &= fD_v D_w s - D_w\left(fD_v s\right) - D_{f[v,w]+w(f)v} s && (22)\\ &= fD_v D_w s - w(f)D_v s - fD_w D_v s - fD_{[v,w]}s - w(f)D_v s && (23)\\ &= f(D_v D_w s - D_w D_v s - D_{[v,w]}s) = fF(v, w)s . && (24) \end{aligned}$$

So, we have shown that $F(v, w)s$ is linear in its $v$ slot, and by antisymmetry also its $w$ slot. Being linear in all slots it is a tensor.

You probably remember the Bianchi identity for the Riemann tensor (curvature tensor on the tangent bundle $TM$),

$$\nabla_{[a} R_{bc]de} = 0 . \tag{25}$$

It turns out that this is true for the connection on any vector bundle,

$$[D_u, [D_v, D_w]] + [D_v, [D_w, D_u]] + [D_w, [D_u, D_v]] = 0\,. \tag{26}$$

However I am not going to ask you to prove this. If you want to see what it takes, I refer you to page 253 of Baez and Muniain's *Gauge theories, knots, and gravity.*

4. Now let's focus on connections on the tangent bundle. Recall that we saw a coordinate calculation of the Lie derivative using a coordinate system's partial derivatives (a valid connection on the tangent bundle). That formula was

$$\mathcal{L}_v T^{i\cdots}{}_{j\ldots} = v^k \partial_k T^{i\cdots}{}_{j\ldots} - T^{k\cdots}{}_{j\ldots}\partial_k v^i - \ldots + T^{i\cdots}{}_{k\ldots}\partial_j v^k + \ldots \tag{27}$$

where there is a correction term with a minus sign for every upstairs index, and one with a minus sign for every downstairs index. Now suppose we have another connection on the tangent bundle, $D$, which is a *symmetric* connection (but we don't necessarily have a metric). Prove that you can use $D$ instead of $\partial$ in Eq. (27) and get the same result.

***Solution:*** Let's invent the notation $\mathcal{L}^{(D)}$ to mean Eq. (27) but expanded with $D$ on the right,

$$\mathcal{L}^{(D)}_v T^{i\cdots}{}_{j\ldots} = v^k D_k T^{i\cdots}{}_{j\ldots} - T^{k\cdots}{}_{j\ldots} D_k v^i - \ldots + T^{i\cdots}{}_{k\ldots} D_j v^k + \ldots\,. \tag{28}$$

Now let's evaluate the difference $\mathcal{L}^{(D)}_v - \mathcal{L}_v$ when acting on some vector $w^i$,

$$\mathcal{L}^{(D)}_v w^i - \mathcal{L}_v w^i = v^k (D_k - \partial_k) w^i - w^k (D_k - \partial_k) v^i\,. \tag{29}$$

Now we let $C^a{}_{bc}$ be the difference between the two connections, $D = \partial + C$, and because these are both symmetric connections, $C$ is symmetric on its lower two indices. Plugging in we find

$$\mathcal{L}^{(D)}_v w^i - \mathcal{L}_v w^i = v^k C^i{}_{kj} w^j - w^k C^i{}_{kj} v^j = 0\,. \tag{30}$$

The cancellation depends crucially on the symmetry in the lower indices of $C$.

Now let's evaluate $\mathcal{L}^{(D)}_v - \mathcal{L}_v$ when acting on one-form $\omega_i$,

$$\begin{aligned} \mathcal{L}^{(D)}_v \omega_i - \mathcal{L}_v \omega_i &= v^k (D_k - \partial_k)\omega_i + \omega_k (D_i - \partial_i) v^k & (31)\\ &= -\, v^k C^j{}_{ki}\omega_j + \omega_k C^k{}_{ij} v^j = 0\,. & (32) \end{aligned}$$

Once again there is a cancellation because of the symmetry of $C$ on the two lower indices.

Finally note that we can extend the above result to tensors of any rank by the Leibniz rule of Lie derivatives. Specifically suppose we evaluate the difference on some tensor product,

$$\begin{aligned} (\mathcal{L}^{(D)}_v - \mathcal{L}_v)(S \otimes T) &= (\mathcal{L}^{(D)}_v S) \otimes T + S \otimes (\mathcal{L}^{(D)}_v T) - (\mathcal{L}_v S) \otimes T - S \otimes (\mathcal{L}_v T) & (33)\\ &= \Big((\mathcal{L}^{(D)}_v - \mathcal{L}_v) S\Big) \otimes T + S \otimes \Big((\mathcal{L}^{(D)}_v - \mathcal{L}_v) T\Big)\,. & (34) \end{aligned}$$

Every tensor can be decomposed into a sum of tensor products of basis vectors and one-forms. Above we saw that $(\mathcal{L}^{(D)}_v - \mathcal{L}_v)$ vanished when acting either on a vector or one-form. Therefore, extending by the Leibniz rule, it vanishes on all tensors, and $\mathcal{L}^{(D)}_v = \mathcal{L}_v$, so we can omit the $(D)$ superscript.

5. Let's apply Frobenius' theorem to the following nonlinear system of PDEs:

$$\begin{aligned} \partial_x f_1 &= A_{11}(x, y, f_1(x,y), f_2(x,y)) & (35a)\\ \partial_y f_1 &= A_{12}(x, y, f_1(x,y), f_2(x,y)) & (35b)\\ \partial_x f_2 &= A_{21}(x, y, f_1(x,y), f_2(x,y)) & (35c)\\ \partial_y f_2 &= A_{22}(x, y, f_1(x,y), f_2(x,y))\,. & (35d) \end{aligned}$$

We want to know what are necessary and sufficient conditions on the $A$ functions for solutions to exist.

(a) To turn this into a geometry problem, we'll want to look for a submanifold in some bigger space (some bundle over $\mathbb{R}^2 \ni (x, y)$). Explain what is this bundle and what are local coordinates for it (thereby stating the dimension of the bundle). What is the dimension of the submanifold we're looking for?

***Solution:*** Over the point $(x, y)$, a fiber of the bundle takes on the pair of values $(f_1(x, y), f_2(x, y))$. The "vertical" direction in the bundle are the two values $f_1$ and $f_2$ of the solution, while the "horizontal" directions are just the coordinates $(x, y)$. So we have a 4-dimensional bundle with total coordinates $(x, y, z, w)$ where $z = f_1(x, y)$ and $w = f_2(x, y)$ would be a solution to the PDE. This is already describing a submanifold by the vanishing of the two functions $\Phi^{(1)}(x, y, z, w) = z - f_1(x, y)$ and $\Phi^{(2)}(x, y, z, w) = w - f_2(x, y)$. Hence the submanifold is co-dimension 2, which is 2-dimensional when we're in a 4-dimensional (bundle) manifold.

(b) Turn the system (35) into a set of vector fields $X_{(i)}$ which define a distribution.

***Solution:*** A more systematic approach would be to use the differential forms approach, but we only covered the vector version of Frobenius' theorem in class, so this will be less rigorous.

Suppose there is a "graph" of $(f_1, f_2)$ on the vertical $(z, w)$ fiber axes over the horizontal $(x, y)$ axes. As we move in the $x$ direction, a vector lying tangent to this graph would be

$$X_{(1)} = \frac{\partial}{\partial x} + \frac{\partial f_1}{\partial x}\frac{\partial}{\partial z} + \frac{\partial f_2}{\partial x}\frac{\partial}{\partial w} \tag{36}$$

$$= \frac{\partial}{\partial x} + A_{11}\frac{\partial}{\partial z} + A_{21}\frac{\partial}{\partial w}\,. \tag{37}$$

Similarly, moving in the $y$ direction, a tangent vector would be

$$X_{(2)} = \frac{\partial}{\partial y} + A_{21}\frac{\partial}{\partial z} + A_{22}\frac{\partial}{\partial w}\,. \tag{38}$$

Indeed notice that the four conditions of these two vectors each being tangent to surface defined by the functions $\Phi^{(1)}$ and $\Phi^{(2)}$ above are identical to the original system of four PDEs.

(c) Using Frobenius' theorem, compute the "integrability conditions", i.e. the necessary and sufficient conditions for existence of solutions, that the $A$'s have to satisfy.

***Solution:*** There is only one bracket to compute,

$$\begin{aligned}[X_{(1)}, X_{(2)}] = {}& (A_{12,x} - A_{11,y} + A_{11}A_{12,z} - A_{12}A_{11,z} + A_{21}A_{12,w} - A_{22}A_{11,w})\frac{\partial}{\partial z} \\ &+ (A_{22,x} - A_{21,y} + A_{11}A_{22,z} - A_{12}A_{21,z} + A_{21}A_{22,w} - A_{22}A_{21,w})\frac{\partial}{\partial w}\,.\end{aligned} \tag{39}$$

Notice that this is purely "vertical." Any non-vanishing linear combination of $X_{(1)}$ and $X_{(2)}$ has some horizontal ($\partial_x$ or $\partial_y$) component. So, the only way for this distribution to be integrable is for this bracket to vanish. Therefore our integrability conditions are

$$0 = A_{12,x} - A_{11,y} + A_{11}A_{12,z} - A_{12}A_{11,z} + A_{21}A_{12,w} - A_{22}A_{11,w} \tag{40}$$

$$0 = A_{22,x} - A_{21,y} + A_{11}A_{22,z} - A_{12}A_{21,z} + A_{21}A_{22,w} - A_{22}A_{21,w}\,. \tag{41}$$

The more pedestrian way to find these is to take the $y$ derivative of the $\partial_x f_1$ equation and demand that it is equal to the $x$ derivative of the $\partial_y f_2$ equation, being sure to properly take the derivatives with the chain rule, e.g.

$$\frac{\partial}{\partial y}\left(A_{11}(x, y, f_1(x, y), f_2(x, y))\right) = \frac{\partial A_{11}(x, y, z, w)}{\partial y} \tag{42}$$

$$+ \frac{\partial A_{11}(x, y, z, w)}{\partial z}\frac{\partial f_1}{\partial y} + \frac{\partial A_{11}(x, y, z, w)}{\partial w}\frac{\partial f_2}{\partial y}$$

$$= A_{11,y} + A_{12}A_{11,z} + A_{22}A_{11,w}\,. \tag{43}$$

A more efficient and index-friendly approach is to rewrite the system to look like a tensor equation,

$$\partial_j f^\alpha = A^\alpha_j(x^i, f^\beta)\,. \tag{44}$$

Now demanding that mixed partials agree means that the antisymmetric part of $\partial_i \partial_j$ should vanish. So, take the $\partial_i$ derivative of Eq. (44) and antisymmetrize, making sure to use the chain rule properly. Then find the system of equations

$$0 = \partial_{[i}\partial_{j]} f^\alpha = A^\alpha_{[j,i]} + \frac{\partial A^\alpha_{[j}}{\partial f^\beta}\frac{\partial f^\beta}{\partial x^{i]}} = A^\alpha_{[j,i]} + \frac{\partial A^\alpha_{[j}}{\partial f^\beta} A^\beta{}_{,i]}\,. \tag{45}$$

This only makes two independent equations: we must have $i = 1$ and $j = 2$ (or swapped), but $\alpha$ can take on either of the two values 1 or 2.