UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
GR II (Phys. 750) — Prof. Leo C. Stein — Spring 2020

Problem Set 3

Due: Tuesday, Feb. 25, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. The FLRW line element in standard coordinates is

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right] \tag{1}$$

where $d\Omega^2 = d\theta^2 + \sin\theta d\phi^2$ is the standard line element on the unit 2-sphere. Here $k = 0$ for a spatially flat universe, otherwise $k = \pm 1$ for a spatially spherical or hyperbolic universe. Consider a congruence with tangent vector $u^a = (\partial_t)^a$.

(a) Show that this congruence is geodesic.

(b) Compute the expansion, shear, and twist of the congruence.

(c) From the Raychaudhuri equation, derive one of the Friedmann equations:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p)\,, \tag{2}$$

if the total stress-energy tensor in the universe is that of a fluid with velocity u^a, energy density (in its own proper reference frame) ρ, and pressure p (again in its own proper reference frame).

2. The Schwarzschild line element in *isotropic coordinates* is

$$ds^2 = -\left(\frac{1 - M/2r}{1 + M/2r} \right)^2 dt^2 + \left(1 + \frac{M}{2r} \right)^4 \left(dr^2 + r^2 d\Omega^2 \right)\,. \tag{3}$$

Note that the r coordinate here is not an areal radius and thus is different from the Schwarzschild r coordinate.

(a) Construct outward-directed unit space-like vector $s^a \propto (\partial_r)^a$ and future-directed unit time-like vector $t^a \propto (\partial_t)^a$.

(b) From these we can construct future-directed null vectors k^a and l^a, with k^a directed outward and l^a inward,

$$k^a = \frac{1}{\sqrt{2}}(t^a + s^a) \tag{4}$$

$$l^a = \frac{1}{\sqrt{2}}(t^a - s^a)\,. \tag{5}$$

Show that k^a is geodesic but is not affinely parametrized. That is, κ is non-vanishing in

$$k^a \nabla_a k^b = \kappa k^b\,. \tag{6}$$

Also find κ (a computer algebra system may be useful and help avoid mistakes).

(c) Construct the projector m_{ab} that projects into the 2-dimensional space orthogonal to k^a and l^a. Use it to compute the expansion of the geodesic congruence k^a,

$$\theta = m^{ab} \nabla_a k_b\,. \tag{7}$$

Compare with $(\nabla_a k^a) - \kappa$ (see problem 2.8 in *A Relativist's Toolkit* for more on this point).